Differential Equations 5410

Sample Midterm Exam 2

Friday, 29 October 2004

Instructions: This in-class exam is 50 minutes. No calculators, notes, tables or books. No answer check is expected.

1. (Undetermined coefficients)
(a) Solve by the method of undetermined coefficients: $y' = 2y + xe^{2x} + x^2$.
(b) Let A be an invertible 2×2 matrix. Find a formula for a particular solution using undetermined coefficients: $\mathbf{u}' = A\mathbf{u} + \mathbf{F}(t)$, $\mathbf{F} = \begin{pmatrix} t^3 - t \\ 1 + t^2 \end{pmatrix}$.
(c) Solve for y_p by the method of undetermined coefficients: $y'' + y = x^2 + x \sin x - cos x$.

2. (Variation of parameters)
(a) State and prove the variation of parameters formula for a second order linear differential equation.
(b) Solve by variation of parameters $y'' - y = xe^x$.
(c) Solve by variation of parameters $\mathbf{u}' = A\mathbf{u} + \mathbf{F}(t)$, given $A = \begin{pmatrix} 1 & 0 \\ 2 & 2 \end{pmatrix}$ and $\mathbf{F}(t) = \begin{pmatrix} t \\ 1 + t \end{pmatrix}$. The answer contains two arbitrary constants c_1, c_2.

3. (Linear 2×2 systems)
(a) Solve $x' = 2x$, $y' = x + y$.
(b) Solve $x' = y$, $y' = x + y$.
(c) Solve $x' = x - y$, $y' = x + y$.

4. (Autonomous equations)
(a) Define *stable equilibrium* and *asymptotically stable equilibrium* for a scalar equation $u' = f(u)$.
(b) State and prove a theorem that says solutions don't cross.
(c) Give an example of an autonomous equation for which solutions cross.
(d) Draw a phase line diagram for $u' = u^3 - u^2$.
(e) Let $f(x, y) = x^2 + y^2 + 2xy$, $g(x, y) = x - 2y$. Determine the linearized system at each possible equilibrium point (x_0, y_0) of the system $x' = f(x, y)$, $y' = g(x, y)$.

5. (Theory of linear systems)
(a) State and prove the superposition principle for a 2×2 linear system $\mathbf{u}' = A\mathbf{u}$.
(b) Prove that the general solution of $\mathbf{u}' = A\mathbf{u} + \mathbf{F}(t)$ is the sum of a particular solution and the general solution of the homogeneous equation. The 2×2 matrix A is constant. Assume continuity for the vector function $\mathbf{F}$.
(c) The Picard iterates for the initial value problem $\mathbf{u}' = A\mathbf{u}$, $\mathbf{u}(0) = \mathbf{u}_0$ can be written out explicitly. Give a general formula and write out the solution according to Picard's limit formula. Assume A is constant 2×2, although the calculation does not depend at all upon the size of A.