Euler, Heun and RK4 Methods
Mathematics 5410

Example. Consider the initial value problem

$$y' = y, \quad y(0) = 1.$$

Compute 20 data points (t_n, y_n) using Euler's method, as follows:

Step 1. Define $t_0 = 0$, $y_0 = 1$, from the initial condition $y(0) = 1$.

Step 2. Define $h = 0.2$, $t_n = nh$, $n = 1, \ldots, 20$.

Step 3. Define $f(t, y) = y$, the right side of the differential equation, and then let
$$y_n = y_{n-1} + hf(t_{n-1}, y_{n-1}), \quad n = 1, \ldots, 20.$$

Plot the resulting approximate solution.

Solution: The algorithm simplifies in this special case to the formulas

$$t_n = nh, \quad y_n = y_{n-1} + hy_{n-1}, \quad n = 1, \ldots, 20.$$

A `maple` implementation appears below.

```
# First group
h:=0.2: t0:=0: y0:=1:
f:=(t,y)->y: L:=[t0,y0]:
# Second group
y1:=y0+h*f(t0,y0): t1:=t0+h:
L:=L,[t1,y1]: t0:=t1: y0:=y1:
# Last group
plot([L]);
```

The list `L` is made by executing the second group twenty times. After the list is completed, then the plot is made by the command on the last line. It is interesting to execute the plot command after each iteration, in order to see the progress made toward computing the approximate solution across $0 \le t \le 4$. A table of values can be made by printing the list `L`. The data set of 21 items:

```
[0, 1], [.2, 1.2], [.4, 1.44], [.6, 1.728],
[.8, 2.0736], [1.0, 2.48832], [1.2, 2.985984],
[1.4, 3.5831808], [1.6, 4.29981696], [1.8, 5.159780352],
[2.0, 6.191736422], [2.2, 7.430083706], [2.4, 8.916100447],
[2.6, 10.69932054], [2.8, 12.83918465], [3.0, 15.40702158],
[3.2, 18.48842590], [3.4, 22.18611108], [3.6, 26.62333330],
[3.8, 31.94799996], [4.0, 38.33759995]
```

Problem 1. Consider the initial value problem

$$y' = y\sin(3t), \quad y(0) = 1.$$

Compute 20 data points (t_n, y_n) using Euler's method, as follows:

Step 1. Define $t_0 = 0$, $y_0 = 1$, from the initial condition $y(0) = 1$.

Step 2. Define $h = 0.2$, $t_n = nh$, $n = 1, \ldots, 20$.

Step 3. Define $f(t,y) = y \sin(3t)$, the right side of the differential equation, and then let
$$y_n = y_{n-1} + hf(t_{n-1}, y_{n-1}), \quad n = 1, \ldots, 20.$$

Plot the resulting approximate solution.

Problem notes. The data set produced should match the one below.

```
[0, 1], [.2, 1], [.4, 1.112928495], [.6, 1.320387066],
[.8, 1.577558229], [1.0, 1.790674729], [1.2, 1.841214735],
[1.4, 1.678259703], [1.6, 1.385713604], [1.8, 1.109633834],
[2.0, .9381367098], [2.2, .8857107226], [2.4, .9408978278],
[2.6, 1.090249902], [2.8, 1.307982259], [3.0, 1.531542301],
[3.2, 1.657777680], [3.4, 1.599978671], [3.6, 1.376021756],
[3.8, 1.106063837], [4.0, .9026966296]
```

Problem 2. Consider the initial value problem

$$y' = y \sin(3t), \quad y(0) = 1.$$

Compute 20 data points (t_n, y_n) using Heun's method, as follows:

Step 1. Define $t_0 = 0$, $y_0 = 1$, from the initial condition $y(0) = 1$.

Step 2. Define $h = 0.2$, $t_n = nh$, $n = 1, \ldots, 20$.

Step 3. Define $f(t,y) = y \sin(3t)$, the right side of the differential equation, and then let
$$z_n = f(t_{n-1}, y_{n-1}),$$
$$y_n = y_{n-1} + 0.5h(z_n + f(t_n, y_{n-1} + hz_n)), \quad n = 1, \ldots, 20.$$

Plot the resulting approximate solution.

Problem notes. The data set produced should match the one below.

```
[0, 1], [.2, 1.056464247], [.4, 1.225702987],
[.6, 1.481558622], [.8, 1.745405020], [1.0, 1.891259348],
[1.2, 1.831894583], [1.4, 1.605296881], [1.6, 1.333344533],
[1.8, 1.118013528], [2.0, 1.005206465], [2.2, 1.006685724],
[2.4, 1.122923829], [2.6, 1.341974103], [2.8, 1.613564575],
[3.0, 1.829323426], [3.2, 1.870194727], [3.4, 1.711265563],
[3.6, 1.447130965], [3.8, 1.198238299], [4.0, 1.035608115]
```

A `maple` implementation for Heun's method appears below, which parallels the example for Problem 1. Left out are the first and third groups, which of course are the same for the example, but differ for Problem 2.

```
# Second group
z1:=f(t0,y0): t1:=t0+h:
```

```
y1:=y0+0.5*h*(z1+f(t1,y0+h*z1)):
L:=L,[t1,y1]: t0:=t1: y0:=y1:
```

Problem 3. Consider the initial value problem

$$y' = y\sin(3t), \quad y(0) = 1.$$

Compute 20 data points (t_n, y_n) using the Runge-Kutta 4 method, as follows:

Step 1. Define $t_0 = 0$, $y_0 = 1$, from the initial condition $y(0) = 1$.

Step 2. Define $h = 0.2$, $t_n = nh$, $n = 1, \ldots, 20$.

Step 3. Define $f(t, y) = y\sin(3t)$, the right side of the differential equation, and then let
$$k_1 = f(t_{n-1}, y_{n-1}),$$
$$k_2 = f(t_{n-1} + h/2, y_{n-1} + hk_1/2),$$
$$k_3 = f(t_{n-1} + h/2, y_{n-1} + hk_2/2),$$
$$k_4 = f(t_{n-1} + h, y_{n-1} + hk_3),$$
$$y_n = y_{n-1} + h(k_1 + 2k_2 + 2k_3 + k_4)/6, \quad n = 1, \ldots, 20.$$

Plot the resulting approximate solution.

Problem notes. The data set produced should match the one below. Check the plot against the exact solution.

```
[0, 1], [.2, 1.059951621], [.4, 1.236829307],
[.6, 1.505420052], [.8, 1.784498524], [1.0, 1.941263717],
[1.2, 1.881857769], [1.4, 1.643372113], [1.6, 1.355473643],
[1.8, 1.129460809], [2.0, 1.013328226], [2.2, 1.016690658],
[2.4, 1.139415643], [2.6, 1.370691595], [2.8, 1.659303389],
[3.0, 1.890826192], [3.2, 1.937762641], [3.4, 1.770721654],
[3.6, 1.488937721], [3.8, 1.223982552], [4.0, 1.053352406]
```

A `maple` implementation for the RK4 method appears below, which parallels the example for Problem 1. Left out are the first and third groups, which of course are the same for the example, but differ for Problem 3.

```
# Second group
k1:=f(t0,y0): t1:=t0+h:
k2:=f(t0+h/2,y0+h*k1/2):
k3:=f(t0+h/2,y0+h*k2/2):
k4:=f(t0+h,y0+h*k3):
y1:=y0+h*(k1+2*k2+2*k3+k4)/6:
L:=L,[t1,y1]: t0:=t1: y0:=y1:
```

Problem 4. Compute $y(1)$ from Euler and RK4 methods using step size $h = 0.05$, for the initial value problem

$$y' = -y^3 + t_3^2, \quad y(0) = 0.$$

Compare the values for percentage error from the true value calculated from `dsolve` in `maple`. Plot on a single graph the two approximate solutions and the one obtained by `dsolve`.

Problem notes. The `dsolve` portion of the solution can be coded in `maple` as follows.

```
t:='t':y:='y':
de:=diff(y(t),t)=-y(t)^3+t^2: ic:=y(0)=0:
ff := dsolve({de,ic},y(t), type=numeric,
               output=listprocedure):
Y := subs(ff,y(t)):
h:=0.05:L1:=seq([i*h,Y(i*h)],i=0..20);
Y(1);
```

The answers:

   Dsolve $Y(1) = .3297007068850505,$

    Euler $Y(1) = .3065826613,$

    RK4 $Y(1) = .3297006477.$

The plot, not produced here, is obtained from three separate plots, using the `maple` syntax `plot({[L1],[L2],[L3]});` to produce all three graphs on one set of axes (two plots appear identical).