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Differential Equations 2280
Sample Midterm Exam 2
Thursday, 30 March 2006

Instructions: This in-class exam is 50 minutes. No calculators, notes, tables or books. No answer check is expected. Details count 75%. The answer counts 25%.

1. (ch3)

(a) Using the *recipe* for higher order constant-coefficient differential equations, write out the general solutions:

1.[25%] $y'' + y' + y = 0$,

2.[25%] $y^{iv} + 4y'' = 0$,

3.[25%] Char. eq. $(r^2 - 3)^2(r^2 + 16)^3 = 0$.

(b) Given $4x''(t) + 4x'(t) + x(t) = 0$, which represents a damped spring-mass system with $m = 4$, $c = 4$, $k = 1$, solve the differential equation [15%] and classify the answer as over-damped, critically damped or under-damped [5%]. Illustrate in a physical model drawing the meaning of constants m, c, k [5%].

Notes on Problem 1.

Part (a)

1: $r^2 + r + 1 = 0$, $y = c_1 y_1 + c_2 y_2$, $y_1 = e^{-x/2}\cos(\sqrt{3}x/2)$, $y_2 = e^{-x/2}\sin(\sqrt{3}x/2)$.

2: $r^{iv} + 4r^2 = 0$, roots $r = 0, 0, 2i, -2i$. Then $y = (c_1 + c_2 x)e^{0x} + c_3\cos 2x + c_4 \sin 2x$.

3: Write as $(r-a)^2(r+a)^2(r^2+16)^3 = 0$ where $a = \sqrt{3}$. Then $y = u_1 e^{ax} + u_2 e^{-ax} + u_3 \cos 4x + u_5 \sin 3x$. The polynomials are $u_1 = c_1 + c_2 x$, $u_2 = c_3 + c_4 x$, $u_3 = c_5 + c_6 x + c_7 x^2$, $u_4 = c_8 + c_9 x + c_{10} x^2$.

Part (b)

Use $4r^2 + 4r + 1 = 0$ and the quadratic formula to obtain roots $r = -1/2, -1/2$. Case 2 of the recipe gives $y = (c_1 + c_2 t)e^{-t/2}$. This is critically damped. The illustration shows a spring, dampener and mass with labels k, c, m, x and the equilibrium position of the mass.

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2. (ch3)
Determine for $y^{iv} - 9y'' = xe^{3x} + x^3 + e^{-3x} + \sin x$ the corrected trial solution for y_p according to the method of undetermined coefficients. Do not evaluate the undetermined coefficients!

Notes on Problem 2.
The homogeneous solution is $y_h = c_1 + c_2 x + c_3 e^{3x} + c_4 e^{-3x}$, because the characteristic polynomial has roots 0, 0, 3, −3.

1 An initial trial solution y is constructed for atoms 1, x, e^{3x}, e^{-3x}, $\cos x$, $\sin x$ giving

$$\begin{aligned}
y &= y_1 + y_2 + y_3 + y_4, \\
y_1 &= (d_1 + d_2 x)e^{3x}, \\
y_2 &= d_3 + d_4 x + d_5 x^2 + d_6 x^3, \\
y_3 &= d_7 e^{-3x}, \\
y_4 &= d_8 \cos x + d_9 \sin x.
\end{aligned}$$

Linear combinations of the listed independent atoms are supposed to reproduce, by assignment of constants, all derivatives of the right side of the differential equation.

2 The fixup rule is applied individually to each of y_1, y_2, y_3, y_4 to give the **corrected trial solution**

$$\begin{aligned}
y &= y_1 + y_2 + y_3, \\
y_1 &= x(d_1 + d_2 x)e^{3x}, \\
y_2 &= x^2(d_3 + d_4 x + d_5 x^2 + d_6 x^3), \\
y_3 &= x(d_7 e^{-3x}), \\
y_4 &= d_8 \cos x + d_9 \sin x.
\end{aligned}$$

The powers of x multiplied in each case are designed to eliminate terms in the initial trial solution which duplicate atoms appearing in the homogeneous solution y_h. The factor is exactly x^s of the Edwards-Penney table, where s is the multiplicity of the characteristic equation root r that produced the related atom in the homogeneous solution y_h. By design, unrelated atoms are unaffected by the fixup rule, and that is why y_4 was unaltered.

3 Undetermined coefficient step skipped, according to the problem statement.

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3. (ch3)
Find by variation of parameters or undetermined coefficients the steady-state periodic solution for the equation $x'' + 2x' + 6x = 5\cos(3t)$.

Notes on Problem 3.
Solve $x'' + 2x' + 6x = 0$ by the recipe to get $x_h = c_1x_1 + c_2x_2$, $x_1 = e^{-t}\cos\sqrt{5}t$, $x_2 = e^{-t}\sin\sqrt{5}t$. Compute the Wronskian $W = x_1x_2' - x_1'x_2 = \sqrt{5}e^{-2t}$. Then for $f(t) = 5\cos(3t)$,

$$x_p = x_1 \int x_2 \frac{-f}{W} dt + x_2 \int x_1 \frac{f}{W} dt.$$

The integrations are horribly difficult, so the method of choice is undetermined coefficients.

The trial solution is $x = d_1\cos 3t + d_2\sin 3t$. Substitute the trial solution to obtain the answers $d_1 = -1/3$, $d_2 = 2/3$. The unique periodic solution x_{SS} is extracted from the general solution $x = x_h + x_p$ by crossing out all negative exponential terms (terms which limit to zero at infinity). If $x_p = d_1\cos 3t + d_2\sin 3t = (1/3)(-\cos 3t + 2\sin 3t)$, then

$$x_{\text{SS}} = \frac{-1}{3}\cos 3t + \frac{2}{3}\sin 3t.$$

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4. (ch5)
The eigenanalysis method says that the system $\mathbf{x}' = A\mathbf{x}$ has general solution $\mathbf{x}(t) = c_1\mathbf{v}_1 e^{\lambda_1 t} + c_2\mathbf{v}_2 e^{\lambda_2 t} + c_3\mathbf{v}_3 e^{\lambda_3 t}$. In the solution formula, $(\lambda_i, \mathbf{v}_i)$, $i = 1, 2, 3$, is an eigenpair of A. Given

$$A = \begin{bmatrix} 4 & 1 & 1 \\ 1 & 4 & 1 \\ 0 & 0 & 4 \end{bmatrix},$$

then
(1) [75%] Display eigenanalysis details for A.
(2) [25%] Display the solution $\mathbf{x}(t)$ of $\mathbf{x}'(t) = A\mathbf{x}(t)$.

Notes on Problem 4.
Answer (1): The eigenpairs are

$$5, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}; \quad 4, \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}; \quad 3, \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}.$$

Answer (2): The eigenanalysis method implies

$$\mathbf{x}(t) = c_1 e^{5t} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + c_2 e^{4t} \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix} + c_3 e^{3t} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}.$$

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5. (ch5)

(a) Find the eigenvalues of the matrix $A = \begin{bmatrix} 1 & 1 & -1 & 0 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 2 & 1 \end{bmatrix}$.

(b) Display the general solution of $\mathbf{u}' = A\mathbf{u}$ according to Putzer's spectral formula. Don't expand matrix products, in order to save time. However, do compute the coefficient functions r_1, r_2, r_3, r_4.

Notes on Problem 5.
(a) Subtract λ from the diagonal elements of A and expand the determinant $\det(A - \lambda I)$ to obtain the characteristic polynomial $(1-\lambda)(1-\lambda)(4-\lambda)(1-\lambda) = 0$. The eigenvalues are the roots: $\lambda = 1, 1, 1, 4$. Used here was the *cofactor rule* for determinants. Sarrus' rule does not apply for 4×4 determinants (an error) and the triangular rule likewise does not directly apply (another error).

(b) Let

$$B = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 4 \end{bmatrix}.$$

Define functions r_1, r_2, r_3, r_4 to be the components of the vector solution $\mathbf{r}(t)$ to the initial value problem

$$\mathbf{r}' = B\mathbf{r}, \quad \mathbf{r}(0) = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

Solving,

$$r_1 = e^t, \quad r_2 = te^t, \quad r_3 = t^2e^t/2,$$
$$r_4 = \frac{1}{27}e^{4t} - \frac{1}{27}e^t - \frac{1}{9}te^t - \frac{1}{6}t^2e^t.$$

Define

$$P_1 = I, \quad P_2 = A - I, \quad P_3 = (A-I)^2, \quad P_4 = (A-I)^3.$$

Then $\mathbf{u} = (r_1P1 + r_2P_2 + r_3P_3 + r_4P_4)\,\mathbf{u}_0$ implies

$$\mathbf{u} = \left(r_1I + r_2(A-I) + r_3(A-I)^2 + r_4(A-I)^3\right)\mathbf{u}_0$$

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