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1. (ch3)

(a) Using the *recipe* for higher order constant-coefficient differential equations, write out the general solutions:

(**a.1**) [25%] $y'' + 4y' + 4y = 0$,

(**a.2**) [25%] $y^{vi} + 4y^{iv} = 0$,

(**a.3**) [25%] Char. eq. $r(r-3)(r^3-9r)^2(r^2+4)^3 = 0$.

(b) Given $6x''(t) + 7x'(t) + 2x(t) = 0$, which represents a damped spring-mass system with $m = 6$, $c = 7$, $k = 2$, solve the differential equation [15%] and classify the answer as over-damped, critically damped or under-damped [5%]. Illustrate in a physical model drawing the meaning of constants m, c, k [5%].

a.1

$r^2 + 4r + 4 = 0$

$(r+2)^2 = 0$

$r = -2, -2$

$$\boxed{y = c_1 e^{-2x} + c_2 x e^{-2x}}$$

a.2

$r^6 + 4r^4 = 0$

$r^4(r^2+4) = 0$

$$\boxed{y = c_1 + c_2 x + c_3 x^2 + c_4 x^3 + c_5 \cos 2x + c_6 \sin 2x}$$

a.3

$r(r-3)\, r^2 (r-3)^2 (r+3)^2 (r^2+4)^3 = 0$

$r^3 (r-3)^3 (r+3)^2 (r^2+4)^3 = 0$

gen sol

$$\begin{cases} y = u_1 e^{0x} + u_2 e^{3x} + u_3 e^{-3x} + u_4 \cos 2x + u_5 \sin 2x \\ u_1 = c_1 + c_2 x + c_3 x^2 \\ u_2 = c_4 + c_5 x + c_6 x^2 \\ u_3 = c_7 + c_8 x \\ u_4 = c_9 + c_{10} x + c_{11} x^2 \\ u_5 = c_{12} + c_{13} x + c_{14} x^2 \end{cases}$$

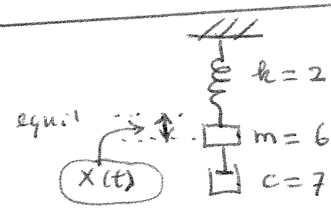
b

$6r^2 + 7r + 2 = 0$

$(2r+1)(3r+2) = 0$

$$\boxed{x(t) = c_1 e^{-t/2} + c_2 e^{-2t/3}}$$

over damped



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2. (ch3)
Determine for $y^{vi} + y^{iv} = x + 2x^2 + x^3 + e^{-x} + x\sin x$ the corrected trial solution for y_p according to the method of undetermined coefficients. Do not evaluate the undetermined coefficients!

$$r^6 + r^4 = 0$$

$$r^4(r^2+1) = 0$$

$$y_h = c_1 + c_2 x + c_3 x^2 + c_4 x^3 + c_5 \cos x + c_6 \sin x$$

- atoms of y_h
  $1, x, x^2, x^3, \cos x, \sin x$

- atoms of deriv of RHS of DE
  $1, x, x^2, x^3, e^{-x}, \cos x, \sin x, x\cos x, x\sin x$

- initial trial sol

$$y = d_1 + d_2 x + d_3 x^2 + d_4 x^3 + d_5 \cos x + d_6 \sin x + d_7 x\cos x + d_8 x \sin x + d_9 e^{-x}$$

- Final trial sol

$$\boxed{y = (d_1 + d_2 x + d_3 x^2 + d_4 x^3)x^4 + (d_5 \cos x + d_6 \sin x + d_7 x\cos x + d_8 x\sin x)x + d_9 e^{-x}}$$

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3. (ch3)
(a) [50%] Find by undetermined coefficients the steady-state periodic solution for the equation $x'' + 4x' + 6x = 10\cos(2t)$.
(b) [50%] Find by variation of parameters a particular solution y_p for the equation $y'' + 3y' + 2y = xe^{2x}$.

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a

$r^2 + 4r + 6 = 0$
$(r+2)^2 + 2 = 0$
$r = -2 \pm \sqrt{2}i$

atoms of $y_h = e^{-2t}\cos\sqrt{2}t,\ e^{-2t}\sin\sqrt{2}t$

- RHS atoms $=$ $\cos 2t$, $\sin 2t$
  No fixup rule
- Final trial sol $y = d_1\cos 2t + d_2 \sin 2t$

Subst: $-4d_1\cos 2t - 4d_2\sin 2t + 4(-2d_1\sin 2t + 2d_2\cos 2t) + 6(d_1\cos 2t + d_2\sin 2t)$
$= 10\cos 2t$

Equations: $\begin{cases} -4d_1 + 8d_2 + 6d_1 = 10 \\ -4d_2 - 8d_1 + 6d_2 = 0 \end{cases}$

$\left(\begin{array}{cc|c} 2 & 8 & 10 \\ -8 & 2 & 0 \end{array}\right) \cong \left(\begin{array}{cc|c} 1 & 4 & 5 \\ 0 & 34 & 40 \end{array}\right) \cong \left(\begin{array}{cc|c} 1 & 4 & 5 \\ 0 & 1 & 20/17 \end{array}\right) \cong \left(\begin{array}{cc|c} 1 & 0 & 5-80/17 \\ 0 & 1 & 20/17 \end{array}\right)$

$\boxed{d_1 = 5/17,\ d_2 = 20/17}$

$\boxed{y = d_1\cos 2t + d_2\sin 2t}$

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b

$r^2 + 3r + 2 = 0$
$(r+2)(r+1) = 0$

$\boxed{y_1 = e^{-x},\ y_2 = e^{-2x}}$

$W = \begin{vmatrix} e^{-x} & e^{-2x} \\ -e^{-x} & -2e^{-2x} \end{vmatrix}$
$= -2e^{-3x} + e^{-3x}$
$= -e^{-3x}$

$y_p = \left(\int \frac{y_2(-f)}{W}\right)y_1 + \left(\int \frac{y_1 f}{W}\right)y_2$
$= \left(\int \frac{-e^{-2x}xe^{2x}}{-e^{-3x}}dx\right)y_1 + \left(\int \frac{e^{-x}xe^{2x}}{-e^{-3x}}dx\right)y_2$
$= \left(\int xe^{3x}dx\right)y_1 + \left(-\int xe^{4x}dx\right)y_2$
$= \left(\frac{x}{3} - \frac{1}{9}\right)e^{3x}e^{-x} + \left(-\left(\frac{x}{4} - \frac{1}{16}\right)e^{4x}\right)e^{-2x}$
$= \left(\frac{x}{3} - \frac{1}{9}\right)e^{2x} + \left(\frac{1}{16} - \frac{x}{4}\right)e^{2x}$
$= \frac{xe^{2x}}{12} - \frac{7}{9(16)}e^{2x}$

$\boxed{y_p = \frac{x}{12}e^{2x} - \frac{7}{144}e^{2x}}$

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use $\int xe^{ax}dx = \left(\frac{x}{a} - \frac{1}{a^2}\right)e^{ax} + C$

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4. (ch5)
The eigenanalysis method says that the system $\mathbf{x}' = A\mathbf{x}$ has general solution $\mathbf{x}(t) = c_1\mathbf{v}_1 e^{\lambda_1 t} + c_2\mathbf{v}_2 e^{\lambda_2 t} + c_3\mathbf{v}_3 e^{\lambda_3 t}$. In the solution formula, $(\lambda_i, \mathbf{v}_i)$, $i = 1, 2, 3$, is an eigenpair of A. Given

$$A = \begin{bmatrix} 5 & 1 & 1 \\ 1 & 5 & 1 \\ 0 & 0 & 7 \end{bmatrix},$$

then
(a) [75%] Display eigenanalysis details for A.
(b) [25%] Display the solution $\mathbf{x}(t)$ of $\mathbf{x}'(t) = A\mathbf{x}(t)$.

$\boxed{a}$

$$\begin{vmatrix} 5-\lambda & 1 & 1 \\ 1 & 5-\lambda & 1 \\ 0 & 0 & 7-\lambda \end{vmatrix} = 0$$

$$(7-\lambda)\left((5-\lambda)^2 - 1\right) = 0$$

$$(7-\lambda)(4-\lambda)(6-\lambda) = 0$$

$$\lambda = 4, 6, 7$$

$\underline{\lambda = 4}$

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 3 & 0 \end{array}\right) \cong \left(\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\begin{cases} x_1 = -t_1 \\ x_2 = t_1 \\ x_3 = 0 \end{cases}$$

$$\boxed{\text{Eigenpair one} = \left(4, \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}\right)}$$

$\underline{\lambda = 6}$

$$\left(\begin{array}{ccc|c} -1 & 1 & 1 & 0 \\ 1 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right) \cong \left(\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\begin{cases} x_1 = t_1 \\ x_2 = t_1 \\ x_3 = 0 \end{cases}$$

$$\boxed{\text{Eigenpair two} = \left(6, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}\right)}$$

$\underline{\lambda = 7}$

$$\left(\begin{array}{ccc|c} -2 & 1 & 1 & 0 \\ 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \cong \left(\begin{array}{ccc|c} 0 & -3 & 3 & 0 \\ 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\cong \left(\begin{array}{ccc|c} -2 & 1 & 1 & 0 \\ 1 & -2 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right) \cong \left(\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\cong \left(\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\boxed{\text{Eigenpair three} = \left(7, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}\right)}$$

$\boxed{b}$

$$\boxed{\vec{x}(t) = c_1 \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} e^{4t} + c_2 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} e^{6t} + c_3 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} e^{7t}}$$

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5. (ch5)

(a) [40%] Find the eigenvalues of the matrix $A = \begin{bmatrix} 4 & 1 & -1 & 0 \\ 1 & 4 & -2 & 1 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 2 & 4 \end{bmatrix}$.

(b) [60%] Display the general solution of $\mathbf{u}' = A\mathbf{u}$ according to Putzer's spectral formula. Don't expand matrix products, in order to save time. However, do compute the coefficient functions r_1, r_2, r_3, r_4. The correct answer for r_4, using λ in increasing magnitude, is $y(x) = -\frac{1}{6}e^{2x} + \frac{1}{2}e^{3x} - \frac{1}{2}e^{4x} + \frac{1}{6}e^{5x}$.

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a

$$\begin{vmatrix} 4-\lambda & 1 & -1 & 0 \\ 1 & 4-\lambda & -2 & 1 \\ 0 & 0 & 2-\lambda & 0 \\ 0 & 0 & 0 & 4-\lambda \end{vmatrix} = (4-\lambda)(2-\lambda)\begin{vmatrix} 4-\lambda & 1 \\ 1 & 4-\lambda \end{vmatrix} = (4-\lambda)(2-\lambda)\left((4-\lambda)^2-1\right)$$

$$= (4-\lambda)(2-\lambda)(3-\lambda)(5-\lambda)$$

$$\boxed{\lambda = 2,3,4,5}$$

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b

$r_1' = 2r_1 \quad r_1(0)=1$

$\boxed{r_1 = e^{2t}}$

$r_2' = 3r_2 + e^{2t} \quad r_2(0)=0$

$(e^{-3t}r_2)' = e^{2t}e^{-3t}$

$e^{-3t}r_2' = -e^{-t}+c$

$r_2 = -e^{2t}+ce^{3t}$

$\boxed{r_2 = e^{3t}-e^{2t}}$

$r_3' = 4r_3 + e^{3t}-e^{2t} \quad r_3(0)=0$

$(e^{-4t}r_3)' = e^{-t}-e^{-2t}$

$e^{-4t}r_3 = -e^{-t}+\frac{1}{2}e^{-2t}+c$

$r_3 = -e^{3t}+\frac{1}{2}e^{2t}+ce^{4t}$

$0 = -1+\frac{1}{2}+c$

$\boxed{r_3 = \frac{1}{2}e^{2t}+\frac{1}{2}e^{4t}-e^{3t}}$

$r_4' = 5r_4 + r_3 \quad r_4(0)=0$

$(e^{-5t}r_4)' = e^{-5t}r_3$

$e^{-5t}r_4 = \int\left(\frac{1}{2}e^{-3t}+\frac{1}{2}e^{-t}-e^{-2t}\right)dt$

$r_4 = e^{5t}\left(-\frac{1}{6}e^{-3t}-\frac{1}{2}e^{-t}+\frac{1}{2}e^{-2t}\right)+ce^{5t}$

$r_4 = -\frac{1}{6}e^{2t}-\frac{1}{2}e^{4t}+\frac{1}{2}e^{3t}+ce^{5t}$

$0 = -\frac{1}{6}-\frac{1}{2}+\frac{1}{2}+c$

$\boxed{r_4 = \frac{1}{6}e^{5t}-\frac{1}{6}e^{2t}-\frac{1}{2}e^{4t}+\frac{1}{2}e^{3t}}$

$$\boxed{\begin{aligned} \vec{u}(t) &= e^{At}\vec{u}(0) \\ e^{At} &= Ir_1 + (A-2I)r_2 + (A-2I)(A-3I)r_3 \\ &\quad + (A-2I)(A-3I)(A-4I)r_4 \end{aligned}}$$

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