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Applied Differential Equations 2280
Midterm Exam 1
Thursday, 16 February 2006

Instructions: This in-class exam is 50 minutes. No calculators, notes, tables or books. No answer check is expected. Details count 75%. The answer counts 25%.

1. (Quadrature Equation)

Solve for the general solution $y(x)$ in the equation $y' = 2\cot x + \dfrac{1250x^3}{1+25x^2} + x\ln(1+x^2)$.

[The required integration talent includes basic formulae, integration by parts, substitution and college algebra.]

$$\int y'\,dx = \int F(x)\,dx$$

$$\boxed{y = I_1 + I_2 + I_3 + C}$$

$$I_1 = \int 2\cot x\,dx = \int \frac{2\cos x}{\sin x}\,dx = \boxed{2\ln|\sin x|}$$

$$I_2 = \int \frac{1250x^3}{1+25x^2}\,dx = \int\left(50x - \frac{50x}{1+25x^2}\right)dx = \boxed{25x^2 - \ln(1+25x^2)}$$

$$I_3 = \int x\ln(1+x^2) = \int \ln(u)\,\frac{du}{2}, \quad u = 1+x^2$$

$$= \frac{1}{2}(u\ln u - u)$$

$$= \boxed{\frac{1}{2}\left((1+x^2)\ln 1+x^2 - 1 - x^2\right)}$$

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2. (Separable Equation Test)
The problem $y' = f(x,y)$ is said to be separable provided $f(x,y) = F(x)G(y)$ for some functions F and G.

(a) [75%] Check ($\boxed{\text{X}}$) the problems that can be put into separable form, but don't supply any details.

☒	$y' = -y(2xy+1) + (2x+3)y^2$ $= -2xy^2 - y + 2xy^2 + 3y^2 = 3y^2 - y$	☐	$yy' = xy^2 + 5x^2y$ Not separable
☒	$y' = e^{x+y} + e^y$ $= e^y(e^x+1)$	☒	$3y' + 5y = 10y^2$ $y' = (10/3)y^2 - (5/3)y$

(b) [25%] State a test which can verify that an equation is not separable. Use the test to verify that $y' = x + \sqrt{|xy|}$ is not separable.

$\boxed{\text{Test}}$:

$y' = f(x,y)$ is not separable if $FG \neq f$ where

$F(x) = \dfrac{f(x,y_0)}{f(x_0,y_0)}$, $G(y) = f(x_0,y)$ and $f(x_0,y_0) \neq 0$

Take $F(x) = \dfrac{f(x,0)}{f(1,0)} = x$ and $G(y) = f(1,y) = 1+\sqrt{|y|}$

Then

$$\begin{aligned} FG &= x(1+\sqrt{|y|}) \\ &= x + x\sqrt{|y|} \\ &\neq x + \sqrt{|xy|} \qquad (\text{e.g., } x=-1, y=1) \end{aligned}$$

So the DE is not separable, by the Test.

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3. (Solve a Separable Equation)

Given $y^2y' = \dfrac{2x^2+3x}{1+x^2}\left(\dfrac{125}{64} - y^3\right)$.

(a) Find all equilibrium solutions.

(b) Find the non-equilibrium solution in implicit form.

To save time, **do not solve** for y explicitly.

(a) $F(x) = \dfrac{2x^2+3x}{1+x^2}$, $G(y) = \dfrac{1}{y^2}\left(\dfrac{125}{64} - y^3\right)$. Then $G(y) = 0$ only for $\boxed{y = \frac{5}{4}}$.

(b) Solve by quadrature $\dfrac{y'}{G(y)} = F(x)$!

$$\int \frac{y^2 y'}{\frac{125}{64} - y^3}\,dx = \int \frac{2x^2+3x}{1+x^2}\,dx$$

$$-\frac{1}{3}\ln\left|\frac{125}{64} - y^3\right| = \int \frac{2(x^2+1) + (3x-2)}{1+x^2}\,dx$$

$$= \int 2\,dx + \frac{3}{2}\int \frac{2x\,dx}{1+x^2} - 2\int \frac{dx}{1+x^2}$$

Then

$$\boxed{-\frac{1}{3}\ln\left|\frac{125}{64} - y^3\right| = 2x + \frac{3}{2}\ln(1+x^2) - 2\arctan(x) + c}$$

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4. (Linear Equations)

(a) [60%] Solve $2v'(t) = -32 + \frac{2}{3t+1}v(t)$, $v(0) = -8$. Show all integrating factor steps.

(b) [30%] Solve $2\sqrt{x+2}\,\frac{dy}{dx} = y$. The answer contains symbol c.

(c) [10%] The problem $2\sqrt{x+2}\,y' = y - 5$ can be solved using the answer y_h from part (b) plus superposition $y = y_h + y_p$. Find y_p. Hint: If you cannot write the answer in a few seconds, then return here after finishing all problems on the exam.

(a) $v' + \frac{-1}{3t+1}v = -16$, $\quad v(0) = -8$

$Q = e^{\int \frac{-dt}{3t+1}}$

$= e^{-\frac{1}{3}\ln|3t+1|}$

$= (3t+1)^{-1/3}$ for $t \cong 0$

$(Qv)'/Q = -16$

$(Qv)' = -16Q$

$Qv = -16\int Q$

$= -16\int (3t+1)^{-1/3}dt$

$= -16\,\frac{(3t+1)^{2/3}}{2/3}\cdot\frac{1}{3} + c$

$= -8(3t+1)^{2/3} + c$

$-8 = -8 + c$ (set $t=0$)

Then $c = 0$ and

$v = -8(3t+1)^{2/3}/Q$

$\boxed{v = -8(3t+1)}$

(b) $y' - \frac{1}{2\sqrt{x+2}}y = 0$

$Q = e^{-\int \frac{dx}{2\sqrt{x+2}}}$

$= e^{-\frac{1}{2}\int (x+2)^{-1/2}dx}$

$= e^{-\frac{1}{2}(x+2)^{1/2}/(1/2)}$

$= e^{-\sqrt{x+2}}$

$(Qy)'/Q = 0$ implies

$y = c/Q$

$\boxed{y = ce^{\sqrt{x+2}}}$

(c) $\boxed{y_p = 5}$ because $y=5$ is an equilibrium solution.

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5. (Stability)

(a) [50%] Draw a phase line diagram for the differential equation

$$dx/dt = 1000\left(2 - \sqrt[5]{x}\right)^3 (2+3x)(9x^2-4)^8.$$

Expected in the diagram are equilibrium points and signs of x' (or flow direction markers $<$ and $>$).

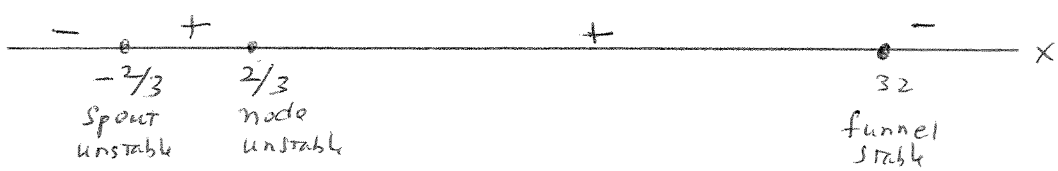
(b) [40%] Draw a phase diagram using the phase line diagram of (a). Add these labels as appropriate: funnel, spout, node, source, sink, stable, unstable. Show at least 8 threaded curves. A direction field is not expected or required.

(c) [10%] Outline how to solve for non-equilibrium solutions, without doing any integrations or long details. [NOT graded]

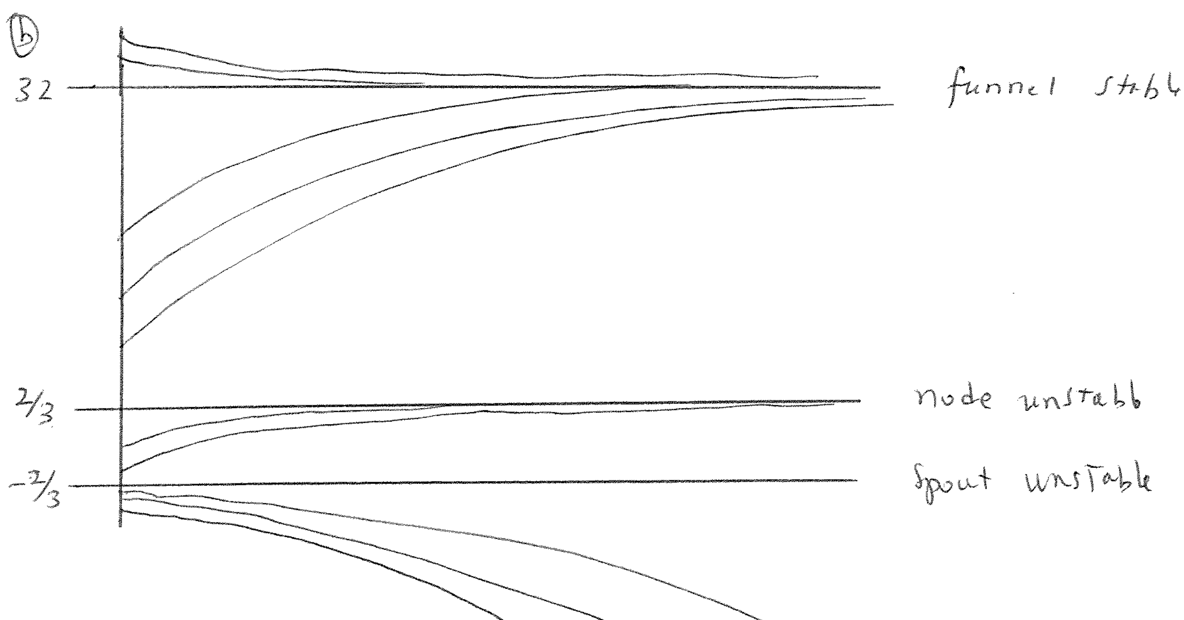
(a) Equilibrium solutions are found from

$$1000\left(2 - x^{1/5}\right)^3 \left(2+3x\right)^9 \left(3x-2\right)^8 = 0$$

$$x = 32,\ 2/3,\ -2/3$$



(b)



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(c) apply the method of quadrature to $\frac{x'}{G(x)} = 1$ where $G(x) = \text{RHS}$ of DE.