Applied Differential Equations 2280
Midterm Exam 1
Thursday, 16 February 2006

Instructions: This in-class exam is 50 minutes. No calculators, notes, tables or books. No answer check is expected. Details count 75%. The answer counts 25%.

1. (Quadrature Equation)

Solve for the general solution $y(x)$ in the equation $y' = 2\cot x + \frac{1250x^3}{1 + 25x^2} + x \ln(1 + x^2)$.

[The required integration talent includes basic formulae, integration by parts, substitution and college algebra.]

\[
\int y' \, dx = \int f(x) \, dx
\]

\[
\frac{y}{y'} = I_1 + I_2 + I_3 + C
\]

\[
I_1 = \int \frac{2}{\cot x} \, dx
\]

\[
= \int \frac{2}{\cot x} \, dx
\]

\[
= \frac{2}{\ln |\csc x|}
\]

\[
I_2 = \int \frac{1250x^3}{1 + 25x^2} \, dx
\]

\[
= \int \frac{50x}{1 + 25x^2} \, dx
\]

\[
= \frac{1}{25} \ln (1 + 25x^2)
\]

\[
I_3 = \int x \ln(1 + x^2) \, dx
\]

\[
= \int \frac{ln(u)}{2} \, du
\]

\[
= \frac{1}{2} (u \ln u - u)
\]

\[
= \frac{1}{2} (1 + x^2) \ln (1 + x^2) - 1 - x^2
\]

Use this page to start your solution. Attach extra pages as needed, then staple.
2. (Separable Equation Test)
   The problem \( y' = f(x, y) \) is said to be separable provided \( f(x, y) = F(x)G(y) \) for some functions \( F \) and \( G \).

   (a) [75%] Check the problems that can be put into separable form, but don’t supply any details.

   \[
   \begin{array}{c|c}
   \times & y' = -y(2xy + 1) + (2x + 3)y^2 \\
   &= -2xy^2 - y + 2xy^2 + 3y^2 - 2y^2 \\
   \times & y' = e^{x+y} + e^y \\
   &= e^y(e^x + 1) \\
   \times & 3y' + 5y = 10y^2 \\
   &= y' = \left(\frac{10}{3}\right)y^2 - \left(\frac{5}{3}\right)y \\
   \end{array}
   \]

   (b) [25%] State a test which can verify that an equation is not separable. Use the test to verify that \( y' = x + \sqrt{|xy|} \) is not separable.

   \[
   \begin{align*}
   F(x) &= \frac{f(x, y)}{f(x_0, y_0)} \\
   G(y) &= f(x, y) \quad \text{and} \quad f(x_0, y_0) \neq 0
   \end{align*}
   \]

   \[
   \begin{align*}
   \text{Take} & \quad F(x) = \frac{f(x, 0)}{f(1, 0)} = x \quad \text{and} \quad G(y) = f(1, y) = 1 + \sqrt{y} \\
   \text{Then} & \quad FG = x \left( 1 + \sqrt{y} \right) \\
   \end{align*}
   \]

   \[
   \begin{align*}
   &= x + x \sqrt{y} \\
   &\neq x + \sqrt{xy} \quad (\text{e.g., } x=-1, y=1)
   \end{align*}
   \]

   So \( y' = x + \sqrt{xy} \) is not separable, by the Test.

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Name. **KEY**

3. (Solve a Separable Equation)

Given \( y^2 y' = \frac{2x^2 + 3x}{1 + x^2} \left( \frac{125}{64} - y^3 \right) \).

(a) Find all equilibrium solutions.

(b) Find the non-equilibrium solution in implicit form.

To save time, do not solve for \( y \) explicitly.

\[(a) \quad F(x) = \frac{2x^2 + 3x}{1 + x^2}, \quad G(y) = \frac{1}{y} \left( \frac{125}{64} - y^3 \right), \quad \text{then} \quad G(y) = 0 \quad \text{implies} \quad y = \frac{5}{4}.
\]

\[(b) \quad \text{Solve } \int y^2 y' \, dx = \int \frac{2x^2 + 3x}{1 + x^2} \, dx.
\]

\[\int \frac{y^2 y'}{6y} \, dx = \int \frac{2x^2 + 3x}{1 + x^2} \, dx\]

\[-\frac{1}{3} \ln \left| \frac{125}{64} - y^3 \right| = \int \frac{2(x^2 + 1) + (3x - 2)}{1 + x^2} \, dx\]

\[= \int 2 \, dx + \frac{3}{2} \int \frac{2x \, dx}{1 + x^2} - 2 \int \frac{dx}{1 + x^2}\]

\[-\frac{1}{3} \ln \left( \frac{125}{64} - y^3 \right) = 2x + \frac{3}{2} \ln (1 + x^2) - 2 \arctan(x) + c\]

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4. (Linear Equations)

(a) [60%] Solve $2v'(t) = -32 + \frac{2}{3t+1}v(t)$, $v(0) = -8$. Show all integrating factor steps.

(b) [30%] Solve $2\sqrt{x} + 2 \frac{dy}{dx} = y$. The answer contains symbol $c$.

(c) [10%] The problem $2\sqrt{x} + 2y' = y - 5$ can be solved using the answer $y_h$ from part (b) plus superposition $y = y_h + y_p$. Find $y_p$. Hint: If you cannot write the answer in a few seconds, then return here after finishing all problems on the exam.

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5. (Stability)
(a) [50%] Draw a phase line diagram for the differential equation

\[ dx/dt = 1000 \left( 2 - \sqrt[3]{x} \right)^3 \left( 2 + 3x \right) \left( 9x^2 - 4 \right)^8. \]

Expected in the diagram are equilibrium points and signs of \( x' \) (or flow direction markers < and >).

(b) [40%] Draw a phase diagram using the phase line diagram of (a). Add these labels as appropriate: funnel, spout, node, source, sink, stable, unstable. Show at least 8 threaded curves. A direction field is not expected or required.

(c) [10%] Outline how to solve for non-equilibrium solutions, without doing any integrations or long details.

Equilibrium solutions are found from

\[ 1000 \left( 2 - \sqrt[3]{x} \right)^3 \left( 2 + 3x \right)^8 \left( 9x^2 - 4 \right)^8 = 0 \]

\[ x = \pm 2, \pm \frac{2}{3}, \pm \frac{2}{3} \]

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Apply the method of quadrature to \( \frac{x'}{G(x)} = 1 \) when \( G(x) = \text{RHS of DE} \).