Applied Differential Equations 2280
Sample Final Exam
Tuesday, 2 May 2006, 4:30-8:00pm

Instructions: This in-class exam is 50 minutes. No calculators, notes, tables or books. No answer check is expected. Details count 75%. The answer counts 25%.

1. **(Quadrature Equation)**

   Solve for the general solution $y(x)$ in the equation $y' = 2\cot x + \dfrac{1250x^3}{1+25x^2} + x\ln(1+x^2)$.

   [The required integration talent includes basic formulae, integration by parts, substitution and college algebra.]

2. **(Separable Equation Test)**
   The problem $y' = f(x,y)$ is said to be separable provided $f(x,y) = F(x)G(y)$ for some functions F and G.

   (a) [75%] Check ($\boxed{\text{X}}$) the problems that can be put into separable form, but don't supply any details.

$\square$ $y' = -y(2xy+1) + (2x+3)y^2$	$\square$ $yy' = xy^2 + 5x^2y$
$\square$ $y' = e^{x+y} + e^y$	$\square$ $3y' + 5y = 10y^2$

   (b) [25%] State a test which can verify that an equation is not separable. Use the test to verify that $y' = x + \sqrt{|xy|}$ is not separable.

3. **(Solve a Separable Equation)**
   Given $y^2y' = \dfrac{2x^2+3x}{1+x^2}\left(\dfrac{125}{64} - y^3\right)$.
   (a) Find all equilibrium solutions.
   (b) Find the non-equilibrium solution in implicit form.
   To save time, **do not solve** for y explicitly.

4. **(Linear Equations)**

   (a) [60%] Solve $2v'(t) = -32 + \dfrac{2}{3t+1}v(t)$, $v(0) = -8$. Show all integrating factor steps.

   (b) [30%] Solve $2\sqrt{x+2}\,\dfrac{dy}{dx} = y$. The answer contains symbol c.

   (c) [10%] The problem $2\sqrt{x+2}\,y' = y - 5$ can be solved using the answer y_h from part (b) plus superposition $y = y_h + y_p$. Find y_p. Hint: If you cannot write the answer in a few seconds, then return here after finishing all problems on the exam.

5. (Stability)

(a) [50%] Draw a phase line diagram for the differential equation

$$dx/dt = 1000\left(2 - \sqrt[5]{x}\right)^3 (2+3x)(9x^2-4)^8.$$

Expected in the diagram are equilibrium points and signs of x' (or flow direction markers $<$ and $>$).

(b) [40%] Draw a phase diagram using the phase line diagram of (a). Add these labels as appropriate: funnel, spout, node, source, sink, stable, unstable. Show at least 8 threaded curves. A direction field is not expected or required.

(c) [10%] Outline how to solve for non-equilibrium solutions, without doing any integrations or long details.

6. (ch3)

(a) Using the *recipe* for higher order constant-coefficient differential equations, write out the general solutions:

(**a.1**) [25%] $y'' + 4y' + 4y = 0$,

(**a.2**) [25%] $y^{vi} + 4y^{iv} = 0$,

(**a.3**) [25%] Char. eq. $r(r-3)(r^3-9r)^2(r^2+4)^3 = 0$.

(b) Given $6x''(t) + 7x'(t) + 2x(t) = 0$, which represents a damped spring-mass system with $m = 6$, $c = 7$, $k = 2$, solve the differential equation [15%] and classify the answer as over-damped, critically damped or under-damped [5%]. Illustrate in a physical model drawing the meaning of constants m, c, k [5%].

7. (ch3)

Determine for $y^{vi} + y^{iv} = x + 2x^2 + x^3 + e^{-x} + x\sin x$ the corrected trial solution for y_p according to the method of undetermined coefficients. Do not evaluate the undetermined coefficients!

8. (ch3)

(a) [50%] Find by undetermined coefficients the steady-state periodic solution for the equation $x'' + 4x' + 6x = 10\cos(2t)$.

(b) [50%] Find by variation of parameters a particular solution y_p for the equation $y'' + 3y' + 2y = xe^{2x}$.

9. (ch5)

The eigenanalysis method says that the system $\mathbf{x}' = A\mathbf{x}$ has general solution $\mathbf{x}(t) = c_1\mathbf{v}_1 e^{\lambda_1 t} + c_2\mathbf{v}_2 e^{\lambda_2 t} + c_3\mathbf{v}_3 e^{\lambda_3 t}$. In the solution formula, $(\lambda_i, \mathbf{v}_i)$, $i = 1, 2, 3$, is an eigenpair of A. Given

$$A = \begin{bmatrix} 5 & 1 & 1 \\ 1 & 5 & 1 \\ 0 & 0 & 7 \end{bmatrix},$$

then

(a) [75%] Display eigenanalysis details for A.

(b) [25%] Display the solution $\mathbf{x}(t)$ of $\mathbf{x}'(t) = A\mathbf{x}(t)$.

10. (ch5)

(a) [40%] Find the eigenvalues of the matrix $A = \begin{bmatrix} 4 & 1 & -1 & 0 \\ 1 & 4 & -2 & 1 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 2 & 4 \end{bmatrix}$.

(b) [60%] Display the general solution of $\mathbf{u}' = A\mathbf{u}$ according to Putzer's spectral formula. Don't expand matrix products, in order to save time. However, do compute the coefficient functions r_1, r_2, r_3, r_4. The correct answer for r_4, using λ in increasing magnitude, is $y(x) = -\frac{1}{6}e^{2x} + \frac{1}{2}e^{3x} - \frac{1}{2}e^{4x} + \frac{1}{6}e^{5x}$.

11. (ch6)
(a) Define *asymptotically stable equilibrium* for $\mathbf{u}' = \mathbf{f}(\mathbf{u})$, a 2-dimensional system.
(b) Give examples of 2-dimensional systems of type saddle, spiral, center and node.
(c) Give a 2-dimensional predator-prey example $\mathbf{u}' = \mathbf{f}(\mathbf{u})$ and explain the meaning of the variables in the model.

12. (ch6)
Find the equilibrium points of $x' = 14x - x^2/2 - xy$, $y' = 16y - y^2/2 - xy$ and classify the linearizations as node, spiral, center, saddle. What classifications can be deduced for the nonlinear system?

13. (ch7)
(a) Define the direct Laplace Transform.
(b) Define Heaviside's unit step function.
(c) Derive a Laplace integral formula for Heaviside's unit step function.
(d) Explain Laplace's Method, as applied to the differential equation $x'(t) + 2x(t) = e^t$, $x(0) = 1$.

14. (ch7)
(a) Solve $\mathcal{L}(f(t)) = \dfrac{100}{s^2+1)(s^2+4)}$ for $f(t)$.
(b) Solve for $f(t)$ in the equation $\mathcal{L}(f(t)) = \dfrac{1}{s^2(s-3)}$.
(c) Find $\mathcal{L}(f)$ given $f(t) = (-t)e^{2t}\sin(3t)$.
(d) Find $\mathcal{L}(f)$ where $f(t)$ is the periodic function of period 2 equal to $t/2$ on $0 \le t \le 2$ (sawtooth wave).

15. (ch7)
(a) Solve $y'' + 4y' + 4y = t^2$, $y(0) = y'(0) = 0$ by Laplace's Method.
(b) Solve $x''' + x'' - 6x' = 0$, $x(0) = x'(0) = 0$, $x''(0) = 1$ by Laplace's Method.
(c) Solve the system $x' = x + y$, $y' = x - y + e^t$, $x(0) = 0$, $y(0) = 0$ by Laplace's Method.

16. (ch9)

(a) Find the Fourier sine and cosine coefficients for the 2-periodic function $f(t)$ equal to $t/2$ on $0 \le t \le 2$.

(b) State Fourier's convergence theorem.

(c) State the results for term-by-term integration and differentiation of Fourier series.

17. (ch10)

(a) Find a steady-state periodic solution by Fourier's method for $x'' + x = F(t)$, where $F(t)$ is 2-periodic and equal to 10 on $0 < t < 1$, equal to -10 on $1 < t < 2$.

(b) Display Fourier's Model for the solution to the heat problem $u_t = u_{xx}$, $u(0,0) = u(1,0) = 0$, $u(x,0) = f(x)$ on $0 \le x \le 1$, $t \ge 0$.

(c) Solve $u_t = u_{xx}$, $u(0,0) = u(\pi,0) = 0$, $u(x,0) = 80\sin^3 x$ on $0 \le x \le \pi$, $t \ge 0$.

18. (ch10)

(a) D'Alembert's solution to the wave equation can be displayed as the superposition of two waves, one moved left and one moving right. Explain this with an example and a snapshot sequence of 4 frames.

(b) Solve by Fourier's Method the plucked string equation $y_{tt} = a^2 y_{xx}$ on $0 < x < 1$, $t \ge 0$, $y(0,t) = y(1,t) = 0$, $y(x,0) = f(x)$, $y_t(x,0) = 0$ with $f(x) = 2(1-x)$. (c) Solve by Fourier's Method the Dirichlet steady-state heat problem $u_{xx} + u_{yy} = 0$, $u(0,y) = u(1,y) = u(x,2) = 0$ on the rectangle $0 \le x \le 1$, $0 \le y \le 2$, with initial data $u(x,0) = f(x)$.