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Prob #: 36 section 5.2
Class: Math 2280-002
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Mixed brine flows from tank 1 into tank 2, from tank 2 into tank 3, and from tank 3 into tank 1, all at the given flow rate $r = 10$ gal/min. The initial amounts $x_1(0) = 18$ lb, $x_2(0) = 0$ lb, $x_3(0) = 0$ lb of salt in the three tanks and their volumes are $V_1 = 20 = V_3$, $V_2 = 50$. First solve for the amounts of salt in the three tanks at time t, then determine the limiting amount (as $t \to +\infty$) of salt in each tank. Finally, construct a figure showing the graphs of $x_1(t)$, $x_2(t)$ and $x_3(t)$.

$$\begin{cases} x_1' = -\frac{1}{2}x_1 + \frac{1}{2}x_2 \\ x_2' = \frac{1}{2}x_1 - \frac{1}{5}x_2 \\ x_3' = \frac{1}{5}x_2 + \frac{1}{2}x_2 \end{cases} \Rightarrow \begin{bmatrix} -\frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{5} & 0 \\ 0 & \frac{1}{5} & -\frac{1}{2} \end{bmatrix} = A$$

D.E. eq. for the probl. also the coefficient matrix

$$\det(A - \lambda I) = \left(-\tfrac{1}{2} - \lambda\right)\left(-\tfrac{1}{5} - \lambda\right)\left(-\tfrac{1}{2} - \lambda\right) + \tfrac{1}{20} =$$ Char. eq.

$$= -\lambda\left(\tfrac{9}{20} + \tfrac{6}{5}\lambda + \lambda^2\right) = 0$$

$\lambda = 0, -\frac{3}{5} + \frac{3}{10}i, -\frac{3}{5} - \frac{3}{10}i$ eigenvalues

$$\begin{bmatrix} -\frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{5} & 0 \\ 0 & \frac{1}{5} & -\frac{1}{2} \end{bmatrix} v = 0 \Rightarrow v = \begin{bmatrix} 4 \\ 10 \\ 4 \end{bmatrix}$$ eigenvector for $\lambda = 0$

$$\begin{bmatrix} -\frac{1}{2} + \frac{3}{5} - \frac{3}{10}i & 0 & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{5} + \frac{3}{5} - \frac{3}{10}i & 0 \\ 0 & \frac{1}{5} & -\frac{1}{2} + \frac{3}{5} - \frac{3}{10}i \end{bmatrix} v = \begin{bmatrix} \frac{1}{10} - \frac{3}{10}i & 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{2}{5} - \frac{3}{10}i & 0 \\ 0 & \frac{1}{5} & \frac{1}{10} - \frac{3}{10}i \end{bmatrix} v = 0 \Rightarrow v = \begin{bmatrix} -\frac{1}{2} - \frac{3}{2}i \\ -\frac{1}{2} + \frac{3}{2}i \\ 1 \end{bmatrix}$$ eigenvector for $\lambda = -\frac{3}{2} + \frac{3}{10}i$

$$x(t) = c\begin{bmatrix} -\frac{1}{2} - \frac{3}{2}i \\ -\frac{1}{2} + \frac{3}{2}i \\ 1 \end{bmatrix} e^{\left(-\frac{3}{5} + \frac{3}{10}i\right)t} + c_0 \begin{bmatrix} 4 \\ 10 \\ 4 \end{bmatrix} e^0 =$$

$$= \begin{bmatrix} -\frac{1}{2} - \frac{3i}{2} \\ -\frac{1}{2} + \frac{3}{2}i \\ 1 \end{bmatrix} e^{-3/5t}\left(\cos\frac{3}{10}\theta + i\sin\frac{3}{10}\theta\right) + c_0\begin{bmatrix} 4 \\ 10 \\ 4 \end{bmatrix} =$$

$$= e^{-3/5t}\begin{bmatrix} -\frac{1}{2}\cos\frac{3}{10}\theta + \frac{3}{2}\sin\frac{3}{10}\theta - \frac{3}{2}i\cos\frac{3}{10}\theta - \frac{1}{2}i\sin\frac{3}{10}\theta \\ -\frac{1}{2}\cos\frac{3}{10}\theta - \frac{3}{2}\sin\frac{3}{10}\theta + \frac{3}{2}i\cos\frac{3}{10}\theta - \frac{1}{2}i\sin\frac{3}{10}\theta \\ \cos\frac{3}{10}\theta + i\sin\frac{3}{10}\theta \end{bmatrix} \Rightarrow$$

$$\begin{cases} x_1 = 4C_0 + \left(-\frac{1}{2}C_1\cos\frac{3}{10}\theta + \frac{3}{2}C_1\sin\frac{3}{10}\theta - \frac{3}{2}C_2\cos\frac{3}{10}\theta - C_2\frac{1}{2}\sin\frac{3}{10}\theta\right)e^{-3/5} \\ x_2 = 10C_0 + \left(-\frac{1}{2}C_1\cos\frac{3}{10}\theta - \frac{3}{2}C_1\sin\frac{3}{10}\theta + \frac{3}{2}C_2\cos\frac{3}{10}\theta - C_2\frac{1}{2}\sin\frac{3}{10}\theta\right)e^{-3/5} \\ x_3 = 4C_0 + \left(C_1\cos\frac{3}{10}\theta + C_2\sin\frac{3}{10}\theta\right)e^{-3/5} \end{cases}$$

General solution

$$\begin{cases} x_1(t) = 4C_0 + e^{-3/5t}\left(\left(-\frac{1}{2}C_1 - \frac{3}{2}C_2\right)\cos\frac{3}{10}t + \left(\frac{3}{2}C_1 - \frac{1}{2}C_2\right)\sin\frac{3}{10}t\right) \\ x_2(t) = 10C_0 + e^{-3/5t}\left(\left(-\frac{1}{2}C_1 + \frac{3}{2}C_2\right)\cos\frac{3}{10}\theta + \left(-\frac{3}{2}C_1 - \frac{1}{2}C_2\right)\sin\frac{3}{10}t\right) \\ x_3(t) = 4C_0 + e^{-3/5t}\left(C_1\cos\frac{3}{10}t + C_2\sin\frac{3}{10}t\right) \end{cases}$$

General solution simplyfied.

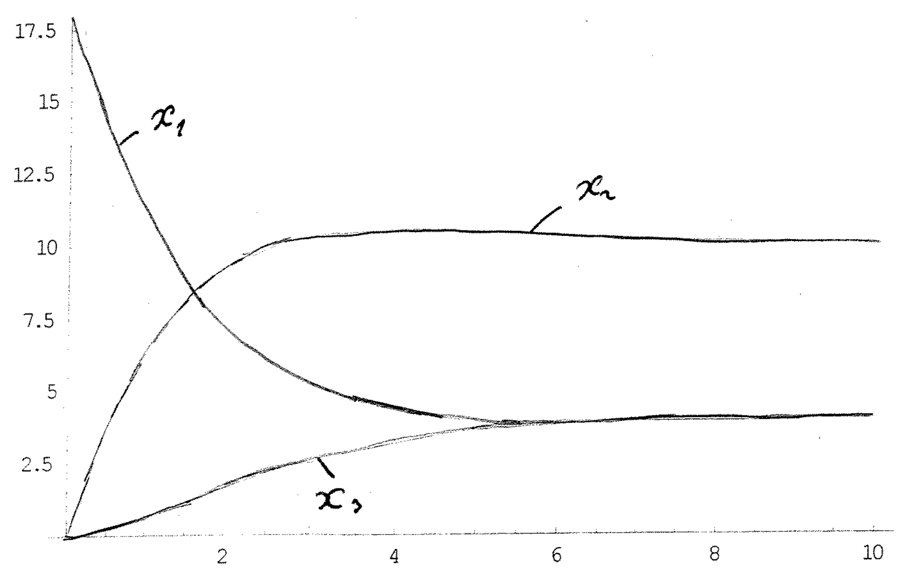
$$\begin{cases} x_1(0) = 4C_0 - \frac{1}{2}C_1 - \frac{3}{2}C_2 = 18 \\ x_2(0) = 10C_0 - \frac{1}{2}C_1 + \frac{3}{2}C_2 = 0 \\ x_3(0) = 4C_0 + C_1 = 0 \end{cases} \Rightarrow \begin{cases} C_0 = 1 \\ C_1 = -4 \\ C_2 = -8 \end{cases}$$

Using initial cond. to find C_0, C_1, C_2

$$\begin{cases} x_1(t) = 4 + e^{-3/5t}\left(14\cos\frac{3}{10}t - 2\sin\frac{3}{10}t\right) \\ x_2(t) = 10 - e^{-3/5t}\left(10\cos\frac{3}{10}t - 20\sin\frac{3}{10}t\right) \\ x_3(t) = 4 - e^{-\frac{3}{5}t}\left(4\cos\frac{3}{10}t + 8\sin\frac{3}{10}t\right) \end{cases}$$

Solution to the prob

$\lim_{t\to\infty} x_1(t) = 4$, $\lim_{t\to\infty}(x_2(t)) = 10$, $\lim_{t\to\infty} x_3(t) = 4$.



Plot of the initial value problem.

Answer Check: D. O. D.