Solution KEY of Best Student Solutions

Name. Toly Zharkikh 2280 Final Exam, 2 May 2006

1. (ch1: Separable Equation)

10 Given $y^2y' = \dfrac{2x^2+4x}{1+x}\left(\dfrac{27}{8} - y^3\right)$.

(a) [20%] Find all equilibrium solutions.

(b) [80%] Find all non-equilibrium solutions in implicit form.

To save time, **do not solve** for y explicitly.

a) $y' = \dfrac{2x^2+4x}{1+x}\left(\dfrac{27}{8}-y^3\right)\dfrac{1}{y^2}$

$y' = 0$ when $y = \frac{3}{2}$ or $x = \cancel{0}, \cancel{-2}$

yes. An equil. sol. is a sol. $y = $ constant. Substitute $y = c$ into the DE, then solve for c.

b) $\displaystyle\int \frac{y^2\,dy}{\frac{27}{8}-y^3} = \int \frac{2x(x+2)}{x+1}\,dx$

$$\begin{array}{r} 2x+2 \\ x+1\overline{\big)\,2x^2+4x} \\ -(2x^2+2x) \\ \hline 2x \\ -(2x+2) \\ \hline -2 \end{array}$$

$-\frac{1}{3}\ln\left(\frac{27}{8}-y^3\right) = \displaystyle\int\left(2x+2-\frac{2}{x+1}\right)dx$

$-\frac{1}{3}\ln\left|\left(\frac{27}{8}-y^3\right)\right| = x^2+2x-2\ln|(x+1)| + c$

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2. (ch1: Linear Equation)

100 (a) [60%] Solve $2v'(t) = -32 + \frac{2}{2t+1}v(t)$, $v(0) = -16$. Show all integrating factor steps.

(b) [40%] Solve $2\sqrt{x+2}\,\frac{dy}{dx} = y + \pi$, $y(0) = -\pi$.

$$v' - \frac{1}{2t+1}v = -16$$

$$\rho = e^{\int \frac{dt}{2t+1}}$$

$$= e^{-\frac{1}{2}\ln(2t+1)}$$

$$= (2t+1)^{-\frac{1}{2}}$$

$$v'\rho - \frac{1}{2t+1}\rho v = -16\rho$$

$$D_t[\rho v] = -16\rho$$

$$\rho v = \int(-16\rho)dt$$

$$\rho v = -16\int(2t+1)^{-\frac{1}{2}}dt$$

$$\rho v = -16(2t+1)^{\frac{1}{2}} + C$$

$$v = -16(2t+1) + C(2t+1)^{\frac{1}{2}}$$

$$v(0) = -16 + C = -16$$

$$C = 0$$

$$v = -16(2t+1) + (\sqrt{2t+1})(c) \quad 0$$

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2b. $dy = (y+\pi)\frac{dx}{2(x+2)^{\frac{1}{2}}}$

$$\int \frac{dy}{y+\pi} = \int 2(x+2)^{-\frac{1}{2}}\,dx$$

$$\ln(y+\pi) = 4(x+2)^{\frac{1}{2}} + C$$

$$y+\pi = c\,e^{4(x+2)^{\frac{1}{2}}}$$

$$y = ce^{4\sqrt{x+2}} - \pi$$

$$y(0) = ce^{4\sqrt{2}} - \pi = -\pi$$

$$c = 0$$

$$y = -\pi$$

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3. (ch3: Recipe)

10 (a) Using the *recipe* for higher order constant-coefficient differential equations, write out the general solutions:

(**a.1**) [25%] $y^{vi} - 9y^{iv} = 0$,

(**a.2**) [25%] Char. eq. $r^2(r-4)(r^3-16r)^2(r^2+16)^3 = 0$.

(b) [50%] Given $4x''(t) + 4x'(t) + x(t) = 0$, which represents a damped spring-mass system with $m = 4$, $c = 4$, $k = 1$, solve the differential equation [40%] and classify the answer as over-damped, critically damped or under-damped [10%].

a1) $r^6 - 9r^4 = 0$

$r^4(r^2-9) = 0$

$r = 0, 0, 0, 0, 3, -3$

$y_h = c_1 + c_2 x + c_3 x^2 + c_4 x^3 + c_5 e^{3x} + c_6 e^{-3x}$

a2) $r = 0, 0, 4, 0, 0, 4, 4, -4, -4, 4i, 4i, -4i, -4i$

$r^2(r-4)(r(r^2-16))^2(r^2+16)^3 = 0$

$y_h = c_1 + c_2 x + c_3 x^2 + c_4 x^3 + c_5 e^{4x} + c_6 x e^{4x} + c_7 x^2 e^{4x} + c_8 e^{-4x} + c_9 x e^{-4x} + c_{10} \sin 4x + c_{11} x \sin 4x + \ldots$

$+ c_{12} \cos 4x + c_{13} x \cos 4x$

b) $4r^2 + 4r + 1 = 0$

$(2r+1)^2 = 0$

$r = -\frac{1}{2}, -\frac{1}{2}$ t

$x_h = c_1 e^{-\frac{1}{2}t} + c_2 x e^{-\frac{1}{2}t}$

$c = 4$

$c_{cr} = \sqrt{4 \cdot 4 \cdot 1} = 4$

critically damped

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4. (ch3: Undetermined Coefficients)

15 Determine for $y^{vi} + 4y^{iv} = x + x^3 + e^{-x} + x\cos 2x$ the corrected trial solution for y_p according to the method of undetermined coefficients. To save time, **do not evaluate** the undetermined coefficients!

$r^6+4r^4=0$

$r^4(r^2+4)=0$

$r=0,0,0,0,2i,-2i$

$y_h=C_1+C_2x+C_3x^2+C_4x^3+C_5\sin 2x+C_6\cos 2x$

atoms of RHS: $1, x, x^2, x^3, e^{-x}, x\cos 2x, x\sin 2x, \cos 2x, \sin 2x$

trial sol: $d_1+d_2x+d_3x^2+d_4x^3+d_5e^{-x}+d_6x\cos 2x+d_7x\sin 2x+d_8\cos 2x+d_9\sin 2x$

corrected trial sol:

$$x^4(d_1+d_2x+d_3x^2+d_4x^3)+d_5e^{-x}+d_6x\cos 2x+d_7x\sin 2x+x^2(d_8\cos 2x+d_9\sin 2x)$$

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5. (ch5: Eigenanalysis Method)
Given

$$A = \begin{bmatrix} 5 & 1 & 1 & 0 \\ 1 & 5 & 1 & 0 \\ 0 & 0 & 7 & 0 \\ 1 & 0 & 5 & 1 \end{bmatrix},$$

then
(a) [75%] Display eigenanalysis details for A.
(b) [25%] Display the solution $\mathbf{x}(t)$ of $\mathbf{x}'(t) = A\mathbf{x}(t)$.

a) $|A-\lambda I| = \begin{vmatrix} 5-\lambda & 1 & 1 & 0 \\ 1 & 5-\lambda & 1 & 0 \\ 0 & 0 & 7-\lambda & 0 \\ 1 & 0 & 5 & 1-\lambda \end{vmatrix} = (1-\lambda)(7-\lambda)\begin{vmatrix} 5-\lambda & 1 \\ 1 & 5-\lambda \end{vmatrix}$

$= (1-\lambda)(7-\lambda)\left[(5-\lambda)^2-1\right]$

$= (1-\lambda)(7-\lambda)\left[25-10\lambda+\lambda^2-1\right]$

$= (1-\lambda)(7-\lambda)(\lambda^2-10\lambda+24)$

$= (1-\lambda)(7-\lambda)(\lambda-6)(\lambda-4)$

$\lambda = 1, 7, 6, 4$

$\boxed{\lambda=1}$ $\left[\begin{array}{cccc|c} 4 & 1 & 1 & 0 & 0 \\ 1 & 4 & 1 & 0 & 0 \\ 0 & 0 & 6 & 0 & 0 \\ 1 & 0 & 5 & 0 & 0 \end{array}\right] \cong \left[\begin{array}{cccc|c} 4 & 1 & 1 & 0 & 0 \\ 0 & 15 & 3 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 \end{array}\right] \cong \left[\begin{array}{cccc|c} 0 & 1 & 5 & 0 & 0 \\ 0 & 5 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{array}\right] \cong \left[\begin{array}{cccc|c} 0 & 0 & 24 & 0 & 0 \\ 0 & -24 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{array}\right]$

$\cong \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$ $\begin{aligned} x_1 &= 0 \\ x_2 &= 0 \\ x_3 &= 0 \\ x_4 &= t \end{aligned}$ so $\vec{v}_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$

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$\boxed{\lambda = 7}$

$$\left[\begin{array}{cccc|c} -2 & 1 & 1 & 0 & 0 \\ {}^{2}1 & {}^{-4}-2 & {}^{2}1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 5 & -6 & 0 \end{array}\right] \cong \left[\begin{array}{cccc|c} -2 & 1 & 1 & 0 & 0 \\ 0 & -3 & 3 & 0 & 0 \\ {}^{2}1 & 0 & {}^{10}5 & {}^{-12}-6 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \cong \left[\begin{array}{cccc|c} -2 & 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & -1 & -11 & +12 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\cong \left[\begin{array}{cccc|c} +2 & -1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & -12 & 12 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \cong \left[\begin{array}{cccc|c} 2 & 0 & -2 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \cong \left[\begin{array}{cccc|c} 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$x_1 = x_4$
$x_2 = x_4$
$x_3 = x_4$
$x_4 = t$

then $\vec{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$

$\boxed{\lambda = 6}$

$$\left[\begin{array}{cccc|c} -1 & 1 & 1 & 0 & 0 \\ 1 & -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 5 & -5 & 0 \end{array}\right] \cong \left[\begin{array}{cccc|c} 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 5 & -5 & 0 \end{array}\right] \cong \left[\begin{array}{cccc|c} 1 & -1 & 0 & 0 & 0 \\ -1 & 0 & -5 & +5 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\cong \left[\begin{array}{cccc|c} 1 & -1 & 0 & 0 & 0 \\ 0 & +1 & +5 & -5 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \cong \left[\begin{array}{cccc|c} 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 5 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \cong \left[\begin{array}{cccc|c} 1 & 0 & 0 & 5 & 0 \\ 0 & 1 & 0 & 5 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$x_1 = -5x_4$
$x_2 = -5x_4$
$x_3 = 0$
$x_4 = t$

then $\vec{v}_3 = \begin{bmatrix} -5 \\ -5 \\ 0 \\ 1 \end{bmatrix}$

$\lambda = 4$

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 \\ 1 & 0 & 5 & -3 & 0 \end{array}\right] \cong \left[\begin{array}{cccc|c} -1 & -1 & -1 & 0 & 0 \\ 1 & 0 & 5 & -3 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \cong \left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 4 & -3 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\cong \left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & -4 & 3 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \cong \left[\begin{array}{cccc|c} 1 & 0 & 4 & -3 & 0 \\ 0 & 1 & 0 & 3 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \cong \left[\begin{array}{cccc|c} 1 & 0 & 0 & -3 & 0 \\ 0 & 1 & 0 & 3 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$x_1 = 3x_4$
$x_2 = -3x_4$
$x_3 = 0$
$x_4 = t$

then $\vec{v}_4 = \begin{bmatrix} 3 \\ -3 \\ 0 \\ 1 \end{bmatrix}$

(b) $x(t) = c_1 v_1 e^{\lambda_1 t} + c_2 v_2 e^{\lambda_2 t} + c_3 v_3 e^{\lambda_3 t} + c_4 v_4 e^{\lambda_4 t}$

$$\boxed{x(t) = c_1 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} e^{t} + c_2 \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} e^{7t} + c_3 \begin{bmatrix} -5 \\ -5 \\ 0 \\ 1 \end{bmatrix} e^{6t} + c_4 \begin{bmatrix} 3 \\ -3 \\ 6 \\ 1 \end{bmatrix} e^{4t}}$$

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6. (ch5; Putzer Spectral Formula) $\lambda_1\ \lambda_2\ \lambda_3\ \lambda_4$ 1∞

Let 4×4 real matrix A have eigenvalues -2, 0, 2, 3.

Display the general solution of $\mathbf{u}' = A\mathbf{u}$ according to Putzer's spectral formula. Don't expand matrix products, in order to save time. However, do compute the coefficient functions r_1, r_2, r_3, r_4. The correct answer for r_4, using λ in increasing magnitude, is $r_4(t) = -\frac{1}{40}e^{-2t} + \frac{1}{12}e^{0t} - \frac{1}{8}e^{2t} + \frac{1}{15}e^{3t}$.

$u' = Au$

$u = e^{At} = r_1 P_1 + r_2 P_2 + r_3 P_3 + r_4 P_4$

$r_1' = r_1 \lambda_1 \qquad r_1(0) = 1$

$r_2' = r_2 \lambda_2 + r_1 \qquad r_2(0) = 0$

$r_3' = r_3 \lambda_3 + r_2 \qquad r_3(0) = 0$

$r_4' = r_4 \lambda_4 + r_3 \qquad r_4(0) = 0$

$P_1 = I$

$P_2 = (I - \lambda A)$

$P_3 = (I - \lambda_2 A)(I - \lambda_1 A)$

$P_4 = (I - \lambda_3 A)(I - \lambda_2 A)(I - \lambda_1 A)$

$\boxed{r_4 = -\frac{1}{40}e^{-2t} + \frac{1}{12} - \frac{1}{8}e^{2t} + \frac{1}{15}e^{3t}}$

$r_4' = \frac{2}{40}e^{-2t} - \frac{2}{8}e^{2t} + \frac{3}{15}e^{3t} = -\frac{3}{40}e^{-2t} + \frac{3}{12} - \frac{3}{8}e^{2t} + \frac{3}{15}e^{3t} - r_3$

$r_3 = -\frac{5}{40}e^{-2t} - \frac{1}{8}e^{2t} + \frac{3}{12} = \boxed{-\frac{1}{8}e^{-2t} - \frac{1}{8}e^{2t} + \frac{1}{4}} = r_3$

$r_3' = \frac{2}{8}e^{-2t} - \frac{2}{8}e^{2t} = -\frac{2}{8}e^{-2t} - \frac{2}{8}e^{2t} + \frac{2}{4} - r_2$

$r_2 = -\frac{4}{8}e^{-2t} + \frac{2}{4} = \boxed{-\frac{1}{2}e^{-2t} + \frac{1}{2}} = r_2$

$r_2' = -\frac{2}{2}e^{-2t} = -r_1$

$\boxed{r_1 = e^{-2t}}$

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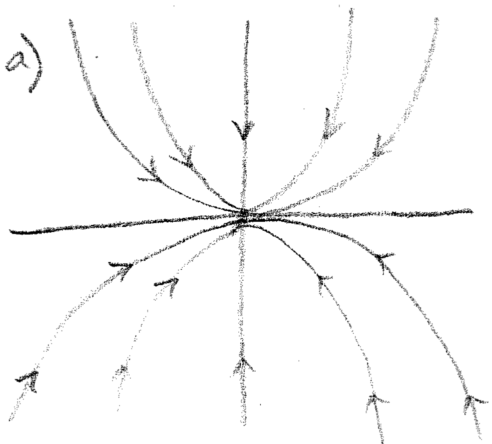
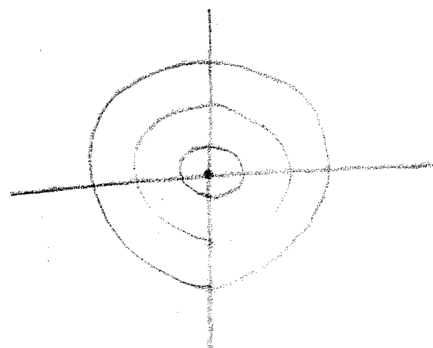
7. (ch6: Stability and Phase Diagram)
(a) [40%] Draw a figure to illustrate an *asymptotically stable equilibrium* for $\mathbf{u}' = \mathbf{f}(\mathbf{u})$, a 2-dimensional system. Draw a second figure to illustrate a *stable equilibrium* that is not asymptotically stable.
(b) [60%] Define saddle, spiral, center and node for a linear 2×2 system $\mathbf{u}' = A\mathbf{u}$, in terms of the eigenvalues of A.

a)



asym. stable node
also possible: a.s. spiral



stable center
not asym.

b) saddle: opposite sign real eigenvales $(\lambda = 1, -2)$

spiral: conjugate roots $(\lambda = 1 \pm 2i)$
with real part $\neq 0$

center: purely imaginary eigenvalues $(\pm i)$

node: real eigenvalues with same sign $(\lambda = 3, 4)$

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8. (ch7: Laplace Theory)

(a) [20%] Display the 4 entries of the basic Laplace Table.

(b) [40%] State 6 Laplace rules.

(c) [40%] State Lerch's Theorem [cancellation law]. Please include hypotheses on the integrands of the Laplace integrals in the cancellation law.

ⓐ $\mathcal{L}(1) = \frac{1}{s}$ $\mathcal{L}(t) = \frac{1}{s^2}$ $\mathcal{L}(t^n) = \frac{n!}{t^{n+1}}$

$\mathcal{L}(e^a) = \frac{1}{s-a}$ $\mathcal{L}(\sin t) = \frac{k}{s^2+k^2}$ $\mathcal{L}(\cos t) = \frac{s}{s^2+k^2}$

---

f + g := functions c := constant

ⓑ ① $\mathcal{L}(f+g) = \mathcal{L}(f) + \mathcal{L}(g)$ ② $\mathcal{L}(cf) = c\,\mathcal{L}(f)$

③ $\mathcal{L}((-t)f(t)) = \frac{d}{ds}\mathcal{L}(f(t))$ ④ $\mathcal{L}(e^{-at}f(t)) = \mathcal{L}(f(t))\Big|_{s\to s+a}$

⑤ $\mathcal{L}(x'(t)) = s\mathcal{L}(x(t)) - x(0)$ ⑥ $F(t) := \mathcal{L}(f(t)) = \int_0^\infty f(t)\, e^{-st}\, dt$

ⓒ $\mathcal{L}(f) = \mathcal{L}(g) \Rightarrow f = g$

if f + g are ~~peicewise~~ of exponential order cont. from $[0,\infty]$

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9. (ch7: Laplace Method)
(a) [50%] Solve for $f(t)$:

$$\mathcal{L}(f) = \frac{d}{ds}\left(\frac{s}{(s+1)^2+4}\right) + \frac{s+2}{s^2+2s+10} + \frac{d}{ds}\left(\frac{s}{(s(s+1)(s-1)}\right)\Bigg|_{s\to s+1}.$$

(b) [25%] Solve $y'' + 4y' + 4y = t$, $y(0) = y'(0) = 0$ by Laplace's Method.
(c) [25%] Solve the system $x' = x - y$, $y' = x + y + 2$, $x(0) = 0$, $y(0) = 0$ by Laplace's Method for $x(t)$ **only**. Don't solve for $y(t)$!

a) $\mathcal{L}(f) = \frac{d}{ds}\left(\frac{s}{(s+1)^2+4}\right) + \frac{s+2}{s^2+2s+10} + \frac{d}{ds}\left(\frac{s}{s(s+1)(s-1)}\right)\Big|_{s\to s+1}$

$$\boxed{\mathcal{L}(f) = T_1 + T_2 + T_3}$$

$$T_1 = \frac{d}{ds}\left(\frac{s}{(s+1)^2+4}\right) = \frac{d}{ds}\left(\frac{s+1}{(s+1)^2+4} + \frac{-1}{(s+1)^2+4}\right) = \frac{d}{ds}\left(\frac{s}{s^2+4}\right)\Big|_{s\to s+1} - \frac{1}{2}\frac{2}{s^2+4}\Big|_{s\to s+1}$$

$$\boxed{T_1 = \mathcal{L}\{(-t)e^{-t}\cos 2t + \tfrac{t}{2}e^{-t}\sin 2t\}}$$

$$T_2 = \frac{s+2}{s^2+2s+10} = \frac{s+2}{(s^2+2s+1)+9} = \frac{s+2}{(s+1)^2+9} = \frac{s+1}{(s+1)^2+9} + \frac{1}{3}\cdot\frac{3}{(s+1)^2+9}$$

$$= \frac{s}{s^2+9}\Big|_{s\to s+1} + \frac{1}{3}\cdot\frac{3}{s^2+9}\Big|_{s\to s+1}$$

$$\boxed{T_2 = \mathcal{L}\{e^{-t}\cos 3t + \tfrac{1}{3}e^{-t}\sin 3t\}}$$

$$T_3 = \frac{d}{ds}\left(\frac{s}{s(s+1)(s-1)}\right)\Big|_{s\to s+1} = \frac{d}{ds}\left(\frac{1}{(s+1)(s-1)}\right)\Big|_{s\to s+1}$$

$$= \frac{d}{ds}\left(\frac{-\frac{1}{2}}{s+1} + \frac{\frac{1}{2}}{s-1}\right)\Big|_{s\to s+1}$$

$$= \mathcal{L}\{(-t)[-\tfrac{1}{2}e^{-t} + \tfrac{1}{2}e^{t}](e^{-t})\}$$

$$\boxed{T_3 = \mathcal{L}\{\tfrac{t}{2}e^{-2t} - \tfrac{t}{2}\}}$$

Then Lerch's thm says $\boxed{f = (-t)e^{-t}\cos 2t + \tfrac{t}{2}e^{-t}\sin 2t + e^{-t}\cos 3t + \tfrac{1}{3}e^{-t}\sin 3t + t/_2\, e^{-2t} - t/_2}$

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(b) $y''+4y'+4y=t \qquad y(0)=y'(0)=0$

$\mathcal{L}(y'')+4\mathcal{L}(y')+4\mathcal{L}(y)=\mathcal{L}(t)$

$s^2\mathcal{L}(y)-\cancel{y'(0)}-\cancel{y(0)}+4s\mathcal{L}(y)-\cancel{y(0)}+4\mathcal{L}(y)=\mathcal{L}(t)$ parts

$(s^2+4s+4)\mathcal{L}(y)=\frac{1}{s^2}$ RHS table forward

$\mathcal{L}(y)=\frac{1}{s^2(s+2)^2}$ isolate $\mathcal{L}(y)$

Partial fractions step:

$$\mathcal{L}(y)=\frac{A}{s}+\frac{B}{s^2}+\frac{C}{s+2}+\frac{D}{(s+2)^2}=\frac{1}{s^2(s+2)^2}$$

$s=0 \qquad B=\frac{1}{4}$

$s=-2 \qquad D=\frac{1}{4}$

$s=1$

$s=-1$

$\frac{1}{(s+2)^2}=A\cancel{s}+B+\frac{C\cancel{s^2}}{s+2}+\frac{D\cancel{s^2}}{\cancel{(s+2)^2}}$

$\frac{1}{s^2}=\frac{\cancel{(s+2)^2}A}{s}+\frac{\cancel{(s+2)^2}B}{s^2}+\cancel{(s+2)}C+D$

$\frac{1}{1(3)^2}=A+B+\frac{C}{3}+\frac{D}{9}=\frac{1}{9}$

$\frac{1}{1(1)}=-A+B+\frac{C}{1}+\frac{D}{1}=1$

$A+\frac{1}{4}+\frac{1}{3}\cdot C+\frac{1}{36}=\frac{1}{9}$

$-A+\frac{1}{4}+C+\frac{1}{4}=1$

$C-A=-\frac{1}{2}$

$C=-\frac{1}{2}+A$

$A+\frac{1}{4}+\frac{1}{3}(-\frac{1}{2}+A)+\frac{1}{36}=\frac{1}{9}$

$A+\frac{1}{4}-\frac{1}{6}+\frac{A}{3}+\frac{1}{36}=\frac{1}{9}$

$\frac{4}{3}A=\frac{1}{9}-\frac{1}{4}+\frac{1}{6}-\frac{1}{36}$

$A=\left[\frac{4}{36}-\frac{9}{36}+\frac{6}{36}-\frac{1}{36}\right]\left(\frac{3}{4}\right)$

$A=0$

$C=-\frac{1}{2}$

Table backwards:

$$\mathcal{L}(y)=\frac{\frac{1}{4}}{s^2}+\frac{-\frac{1}{2}}{s+2}+\frac{\frac{1}{4}}{(s+2)^2}$$

$$\mathcal{L}(y)=\frac{1}{4}\left(\frac{1}{s^2}\right)+-\frac{1}{2}\left(\frac{1}{s+2}\right)+\frac{1}{4}\left(\frac{1}{s^2}\right)\Big|_{s+2}$$

$$=\frac{1}{4}\mathcal{L}(t)-\frac{1}{2}\mathcal{L}(e^{-2t})+\frac{1}{4}\mathcal{L}(e^{-2t}\cdot t)$$

$$\boxed{\begin{array}{l}\text{Lerch's thm:}\\ y=\frac{1}{4}(t)-\frac{1}{2}(e^{-2t})+\frac{1}{4}(e^{-2t}\cdot t)\end{array}}$$

c) $x' = x - y$ $\quad x(0) = 0$

$y' = x + y + 2$ $\quad y(0) = 0$

$$\mathcal{L}(x') = \mathcal{L}(x) - \mathcal{L}(y) \qquad \mathcal{L}(y') = \mathcal{L}(x) + \mathcal{L}(y) + \mathcal{L}(2)$$

$$s\mathcal{L}(x) = \mathcal{L}(x) - \mathcal{L}(y) \qquad s\mathcal{L}(y) = \mathcal{L}(x) + \mathcal{L}(y) + \mathcal{L}(2)$$

$$(s-1)\mathcal{L}(x) + \mathcal{L}(y) = 0 \qquad (-1)\mathcal{L}(x) + (s-1)\mathcal{L}(y) = \frac{2}{s}$$

$$\left|\begin{array}{cc|c} s-1 & 1 & 0 \\ -1 & s-1 & 2/s \end{array}\right|$$

$$\mathcal{L}(x) = \frac{\begin{vmatrix} 0 & 1 \\ 2/s & s-1 \end{vmatrix}}{\begin{vmatrix} s-1 & 1 \\ -1 & s-1 \end{vmatrix}} = \frac{-\frac{2}{s}}{(s-1)^2+1}$$

Kramer's Rule

$$\mathcal{L}(x) = \frac{-2/s}{(s-1)^2+1} = \frac{-2}{s[(s-1)^2+1]} \cancel{= \frac{-2}{s(s^2-2s+2)}}$$

partial fractions

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9. (ch7: Laplace Method)

(a) [50%] Solve for $f(t)$:

$$\mathcal{L}(f) = \frac{d}{ds}\left(\frac{s}{(s+1)^2+4}\right) + \frac{s+2}{s^2+2s+10} + \frac{d}{ds}\left(\frac{s}{(s)(s+1)(s-1)}\right)\Bigg|_{s\to s+1}.$$

(b) [25%] Solve $y''+4y'+4y=t$, $y(0)=y'(0)=0$ by Laplace's Method.

(c) [25%] Solve the system $x'=x-y$, $y'=x+y+2$, $x(0)=0$, $y(0)=0$ by Laplace's Method for $x(t)$ **only**. Don't solve for $y(t)$!

let $f(t)=f_1(t)+f_2(t)+f_3(t)$ break up problem into 3 parts

a)

$$\mathcal{L}(f_1)=\frac{d}{ds}\left(\frac{s-1}{s^2+4}\right)\Bigg|_{s\to s+1}$$

$$=\frac{d}{ds}\left(\frac{s}{s^2+4}-\frac{1}{2}\frac{2}{s^2+4}\right)\Bigg|_{s\to s+1}$$

$$\mathcal{L}(f_1(t))=\frac{d}{ds}\mathcal{L}\left(\cos 2t-\frac{1}{2}\sin 2t\right)$$

$$=\cancel{\frac{d}{ds}}\mathcal{L}\left(e^{-t}\left(\cos 2t-\frac{1}{2}\sin 2t\right)\right)$$

$f_1(t)= e^{-t}\left(\cos 2t-\frac{1}{2}\sin 2t\right)$ would be correct for shift Thm only

$f_1(t)= -te^{-t}\left(\cos 2t-\frac{1}{2}\sin 2t\right)$ Correct for s-diff Theorem and shift Thm

$$\mathcal{L}(f_2)=\frac{s+2}{(s+1)^2+9}$$

$$=\frac{s+1}{s^2+9}\Bigg|_{s\to s+1}$$

$$=\left(\frac{s}{s^2+9}+\frac{1}{3}\frac{3}{s^2+9}\right)\Bigg|_{s\to s+1}$$

$f_2(t)=\left(\cos 3t+\frac{1}{3}\sin 3t\right)e^{-t}$ ✓

$$\mathcal{L}(f_3)=\frac{d}{ds}\left(\frac{1}{2}\frac{1}{s+1}+\frac{1}{2}\frac{1}{s-1}\right)\Bigg|_{s\to s+1}$$

$$=\frac{d}{ds}\left(\frac{1}{2}\frac{1}{s+2}+\frac{1}{2}\frac{1}{s}\right)$$

$$\mathcal{L}(f_3(t))=\frac{d}{ds}\mathcal{L}\left(\frac{1}{2}e^{-2t}+\frac{1}{2}\right)$$

$$f_3(t)=(-t)\left(\frac{1}{2}e^{-t}+\frac{1}{2}e^{t}\right)e^{-t}$$

$$=-\frac{t}{2}\left(e^{-2t}+1\right)$$

Use this page to start your solution. Attach extra pages as needed, then staple.

#9 b) $\mathcal{L}(y'') + 4\mathcal{L}(y') + 4\mathcal{L}(y) = \mathcal{L}(t)$

$$s^2\mathcal{L}(y) + 4s\mathcal{L}(y) + 4\mathcal{L}(y) = \frac{1}{s^2}$$

$$(s^2+4s+4)\mathcal{L}(y) = \frac{1}{s^2}$$

$$\mathcal{L}(y) = \frac{1}{s^2(s^2+4s+4)}$$

$$= \frac{1}{s^2(s+2)^2}$$

$$= \frac{a}{s} + \frac{b}{s^2} + \frac{c}{s+2} + \frac{d}{(s+2)^2}$$

$b = \frac{1}{4}$ $d = \frac{1}{4}$

one way to find a, c

$$1 = as(s+2)^2 + \frac{1}{4}(s+2)^2 + cs^2(s+2) + \frac{s^2}{4}$$

$s=1$: $1 = 9a + \frac{9}{4} + 3c + \frac{1}{4}$

$s=-1$: $1 = -a + \frac{1}{4} + c + \frac{1}{4}$

Then $a = -1/4$, $c = 1/4$.

$$= \mathcal{L}\left(-\frac{1}{4} + \frac{1}{4}t + \frac{1}{4}e^{-2t} + \frac{1}{4}te^{-2t}\right)$$

$$\boxed{y(t) = \frac{1}{4}\left(-1 + t + e^{-2t} + te^{-2t}\right)}$$

c) $\mathcal{L}(x') = \mathcal{L}(x) - \mathcal{L}(y)$

$$s\mathcal{L}(x) - \mathcal{L}(x) + \mathcal{L}(y) = 0$$

$$(s-1)\mathcal{L}(x) + \mathcal{L}(y) = 0$$

$\mathcal{L}(y') = \mathcal{L}(x) + \mathcal{L}(y) + \mathcal{L}(2)$

$$s\mathcal{L}(y) - \mathcal{L}(y) - \mathcal{L}(x) = \frac{2}{s}$$

$$(s-1)\mathcal{L}(y) - \mathcal{L}(x) = \frac{2}{s}$$

should be $\begin{bmatrix} 0 \\ 2/s \end{bmatrix}$

$$\mathcal{L}(x) = \frac{\begin{vmatrix} \frac{2}{s} & 1 \\ 0 & s-1 \end{vmatrix}}{\begin{vmatrix} s-1 & 1 \\ -1 & s-1 \end{vmatrix}}$$

Cramer's Rule applied

$\begin{bmatrix} 0 \\ 2/s \end{bmatrix}$

$$\frac{\begin{vmatrix} s-1 & \frac{2}{s} \\ -1 & 0 \end{vmatrix}}{\begin{vmatrix} s-1 & 1 \\ -1 & s-1 \end{vmatrix}}$$

(not requested)

Answer

$$\boxed{\mathcal{L}(x) = \frac{-2}{s((s-1)^2+1)} \qquad x = -1 + e^t(\cos t - \sin t)}$$

$$= \frac{2s-2}{(s-1)^2+1}$$

$$= \frac{s}{s^2+1}\Big|_{s \to s-1}$$

$$x = e^t \cos t$$

$$= \frac{+\frac{2}{s}}{(s-1)^2+1} = \frac{2}{s((s-1)^2+1)}$$

$$y = \mathcal{L}^{-1}\left(\frac{2}{s(s-1)^2+s}\right)$$

Name. Clayton Fowler 2280 Final Exam, 2 May 2006

10. (ch9: Fourier Series)

(a) [30%] Find the Fourier sine and cosine coefficients for the 1-periodic function $f(t)$ equal to $t-1$ on $0 \le t \le 1$.

(b) [20%] Define a function f piecewise continuous on $a < t < b$.

(c) [20%] State Fourier's convergence theorem.

(d) [30%] Find the (standard) Fourier coefficients for the 2π-periodic function $f(t) = \sin^2(t) - \cos(2t) + \sin(2t)\cos(3t)$.

a) $t-1$

$u = t \quad dv = \cos(n\pi t)$

$du = dt \quad v = \dfrac{\sin(n\pi t)}{n\pi}$

$$a_n = \frac{1}{2}\int_0^1 (t-1)\cos(n\pi t)\,dt$$

$$= \frac{1}{2}\left(\int_0^1 t\cos(n\pi t)\,dt - \int_0^1 \cos(n\pi t)\,dt\right)$$

$$= \frac{1}{2}\left(\frac{t\sin(n\pi t)}{n\pi} - \int_0^1 \frac{\sin(n\pi t)}{n\pi}\,dt - \frac{\sin(n\pi t)}{n\pi}\Big|_0^1\right)$$

$$= \frac{1}{2}\left(\frac{t\sin(n\pi t)}{n\pi} + \frac{\cos(n\pi t)}{n^2\pi^2} - \frac{\sin(n\pi t)}{n\pi}\Big|_0\right)$$

$$= \frac{-1}{2n^2\pi^2} + \cancel{\cos}\frac{\cos(n\pi)}{2n^2\pi^2} = \frac{(-1)^n - 1}{4n^2\pi^2} = \boxed{a_n = \frac{-1}{n^2\pi^2} \text{ for odd } n \quad 0 \text{ for even } n}$$

$$a_0 = \frac{1}{2}\int_0^1 (t-1)\,dt = \frac{t^2}{2} - t\Big|_0 = -\frac{1}{4} = a_0$$

$$b_n = \frac{1}{2}\int_0^1 (t-1)\sin(n\pi t) = \frac{1}{2}\int_0^1 t\sin(n\pi t)\,dt - \frac{1}{2}\int_0^1 \sin(n\pi t)\,dt$$

$$= \frac{-t\cos(n\pi t)}{2n\pi} + \frac{\sin(n\pi t)}{2n^2\pi^2} + \frac{1}{2}\frac{\cos(n\pi t)}{2n\pi}\Big|_0^1 =$$

Use this page to start your solution. Attach extra pages as needed, then staple.

$$b_n = \frac{-1}{4n\pi} - \frac{2\cos(n\pi)}{4n\pi} + \frac{\cos(n\pi)}{4n\pi} = \frac{-1 - \cos(n\pi)}{4n\pi} = \text{0 for } n \text{ odd}$$

$$\text{for } n \quad = \frac{-1}{2n\pi}$$

b) a function is piecewise continuous on ~~[~~ (a,b) if it is continuous ~~on a~~ everywhere everywhere on (a,b) except ~~other~~ at a finite number of places

c) If a periodic function is piecewise continuous over its period, then it converges to $f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(\frac{n\pi}{p} t) + b_n \sin(\frac{n\pi}{p} t)$ at all the points of continuity, and to the average of the limits of the right and left at the place of discontinuity

d) $f(t) = \sin^2(t) - \cos 2t + \sin 2t \cos(3t)$ 2π periodic

$\sin 2t \cos 3t = \frac{\sin(5t) - \sin(t)}{2}$

$\rightarrow b_5 = 1/2$

$b_1 = -1/2$

$$\sin^2(t) = \frac{1}{2} - \frac{\cos(2t)}{2}$$

$$\rightarrow f(t) = \frac{1}{2} - \frac{3}{2}\cos(2t) + \frac{\sin(5t)}{2} - \frac{\sin(t)}{2}$$

$$\rightarrow \boxed{\begin{aligned} a_0 &= 1 \\ a_2 &= -\frac{3}{2} \\ b_5 &= \frac{1}{2} \\ b_1 &= -\frac{1}{2} \end{aligned}}$$

all other coefficients are zero

Name. Toly Zharkikh

2280 Final Exam, 2 May 2006

10. (ch9: Fourier Series)

(a) [30%] Find the Fourier sine and cosine coefficients for the 1-periodic function $f(t)$ equal to $t-1$ on $0 \le t \le 1$.

(b) [20%] Define a function f piecewise continuous on $a < t < b$.

(c) [20%] State Fourier's convergence theorem.

(d) [30%] Find the (standard) Fourier coefficients for the 2π-periodic function $f(t) = \sin^2(t) - \cos(2t) + \sin(2t)\cos(3t)$.

a) $A(n)$ $a_n = \frac{1}{\pi}\int_0^1 (t-1)\sin nt\, dt$

Replace π by $1/2$

$= \frac{1}{\pi}\int_0^1 (t\sin nt - \sin nt)\,dt$

v du

$dv = dt \quad u = -\frac{1}{n}\cos nt$

$= \frac{1}{\pi}\left(-\frac{t}{n}\cos nt - \frac{1}{n}\int_0^1 \cos nt\, dt - \int_0^1 \sin nt\, dt\right)$

$= \frac{-1}{\pi n}\left[t\cos nt + \frac{1}{n}\sin nt + \cos nt\right]_0^1$

$= \frac{-1}{\pi n}\left(2\cos n + \frac{1}{n}\sin n - 1\right)$

ans: $a_n = A(2n\pi)$
$a_0 = -1/2$

$B(n)$ $b_n = \frac{1}{\pi}\int_0^1 (t-1)\cos nt\, dt$

$= \frac{1}{\pi}\int_0^1 (t\cos nt - \cos nt)\,dt$

$dv = dt \quad u = \frac{1}{n}\sin nt$

$= \frac{1}{\pi}\left(\frac{t}{n}\sin nt - \frac{1}{n}\int_0^1 \sin nt\,dt - \int_0^1 \cos nt\, d\right.$

$= \frac{1}{\pi n}\left[t\sin nt + \cos nt - \sin nt\right]_0^1$

$= \frac{1}{\pi n}\left(\sin n + \cos n - \sin n - 1\right)$

$= \frac{1}{\pi n}(\cos n - 1)$

ans: $b_n = B(2n\pi)$

b) $f = \begin{cases} 1 & x = 0 \\ 0 & x \neq 0 \end{cases}$

$f(t) = \begin{cases} f_1(t) & a < t < t_1 \\ \vdots & \vdots \\ f_n(t) & t_{n-1} < t < b \end{cases}$

where $f_i(t)$ is continuous on (a,b), $i = 1, 2, \ldots, n$.

a) The orthogonal set of functions on $[0,1]$ is

$\{\cos(2n\pi t)\}_{n=0}^{\infty}$ plus

$\{\sin(2n\pi t)\}_{n=1}^{\infty}$

Then $\int_0^1 \sin^2(2n\pi t)\,dt = \int_0^1 \cos^2(2n\pi t)\,dt = \frac{1}{2}$

except for $n = 0$, where $\int_0^1 1\cdot dt = 1$

Use this page to start your solution. Attach extra pages as needed, then staple.

c) Let f be piecewise continuously differentiable on $[-\pi, \pi]$. Then the Fourier series of f converges to $f(t)$ at points of continuity of f and to $[f(t+) + f(t-)]/2$ otherwise.

10 d)

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} (\sin^2 t - \cos 2t + \sin 2t \cos 3t) \sin nt \, dt$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} (\sin^2 t - \cos 2t + \sin 2t \cos 3t) \cos nt \, dt$$

d) Evaluate using trig identities

$$\sin^2 t = \frac{1 - \cos 2t}{2}$$

$$2 \sin 2t \cos 3t = \sin 5t - \sin t$$ from $\sin a \pm b = \sin a \cos b \pm \sin b \cos a$

Then

$$\sin^2 t - \cos 2t + \sin 2t \cos 3t = \frac{1}{2} - \frac{3}{2}\cos 2t + \frac{1}{2}\sin 5t - \frac{1}{2}\sin t$$

$= \text{finite Fourier series}$

No integration are required and it is unnecessary to use formulae for a_n, b_n.