Eigenanalysis

The subjects of eigenvalues and eigenvectors were created in order to <u>simplify</u> the equations which represent a physical problem. The equations can be algebraic equations, ordinary or partial differential equations, integral equations or infinite series expansions.

The simplifications are realized by finding a <u>special set of coordinates</u> for the problem, which transforms complexity out of the problem's equations. Some examples:

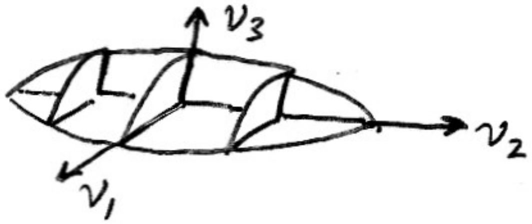
- Geometric problems
- Chemical kinetics
- Heat transfer
- Electrical networks
- Coupled mechanical systems

- In differential equations, eigenanalysis is used <u>to find easy-to-solve equations</u> related to the original problem by a change of variables.

- In geometry, eigenanalysis is used <u>to determine coordinate systems</u> which greatly simplify and standardize the coorresponding algebraic equations for the geometric objects.

- In heat transfer, eigenanalysis is used <u>to determine a basis of functions</u>, by which the temperature function can be simply [illegible]

A Geometry Problem: the Ellipsoid



an ellipsoid in $\mathbb{R}^3$

Eigenanalysis for an ellipsoid calculates the semiaxis directions v_1, v_2, v_3 and the semiaxis lengths ℓ_1, ℓ_2, ℓ_3.

- In the coordinate system created by v_1, v_2, v_3, the ellipsoid equation takes the standard simplified form

(1) $$\frac{x^2}{\ell_1^2} + \frac{y^2}{\ell_2^2} + \frac{z^2}{\ell_3^2} = 1 .$$

- The role of eigenanalysis is to calculate from a given ellipsoid equation, e.g.,

(2) $$x^2 + 2y^2 + 4z^2 + xz - xy = 10,$$

The hidden coordinate system v_1, v_2, v_3 and the hidden values $\frac{1}{\ell_1^2}, \frac{1}{\ell_2^2}, \frac{1}{\ell_3^2}$, which transforms (2) into (1).

- Briefly, eigenanalysis is a computational tool to find a change of variables to convert (2) into (1), hence removing the algebraic complexity from the ellipsoid's equation.

The Eigenanalysis of Matrices

Eigenvalue (= hidden Value or proper Value)

A number λ, real or complex, is called an eigenvalue of an $n \times n$ matrix A provided the equation

$$A \vec{X} = \lambda \vec{X}$$

has a solution $\vec{X} \neq \vec{0}$.

Theorem [how to find eigenvalues]
An eigenvalue λ satisfies $(A - \lambda I)\vec{X} = \vec{0}$, for some $\vec{X} \neq \vec{0}$. Cramer's rule says that all eigenvalues are determined from the polynomial equation

$$\det(A - \lambda I) = 0.$$

Eigenvector (= hidden vector or proper vector)

A given matrix A of size $n \times n$ has eigenpair $(\lambda, \vec{X})$ provided ⓐ $\vec{X} \neq \vec{0}$, ⓑ $A\vec{X} = \lambda \vec{X}$. The value λ is an eigenvalue and the vector $\vec{X}$ is an eigenvector for λ.

Theorem [how to find Eigenvectors]
The independent eigenvectors that correspond to eigenvalue λ of A are exactly the basis of solutions

Let $A = \begin{bmatrix} 1 & -1 \\ -4 & 1 \end{bmatrix}$. Find the hidden vectors and hidden values of A.

Hidden Values

$$\det(A - \lambda I) = \begin{vmatrix} 1-\lambda & -1 \\ -4 & 1-\lambda \end{vmatrix}$$

Hidden values λ satisfy $\det(A - \lambda I) = 0$

$$= (1-\lambda)^2 - 4$$

Expand

$$= (\lambda+1)(\lambda-3)$$

Factor

$$\boxed{\lambda_1 = -1, \ \lambda_2 = 3}$$

Hidden values found.

Hidden Vectors

$$A - \lambda_1 I = \begin{bmatrix} 1-\lambda_1 & -1 \\ -4 & 1-\lambda_1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & -1 \\ -4 & 2 \end{bmatrix}$$

$$\cong \begin{bmatrix} 2 & -1 \\ 0 & 0 \end{bmatrix}$$

RREF found; solution: $x = \frac{1}{2}t$, $y = t$, $-\infty < t < \infty$

$$\boxed{X_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}}$$

First hidden vector (take $t = 2$).

$$A - \lambda_2 I = \begin{bmatrix} 1-\lambda_2 & -1 \\ -4 & 1-\lambda_2 \end{bmatrix}$$

$$= \begin{bmatrix} -2 & -1 \\ -4 & -2 \end{bmatrix}$$

$$\cong \begin{bmatrix} 2 & 1 \\ 0 & 0 \end{bmatrix}$$

RREF found; solution: $x = -\frac{1}{2}t$, $y = t$, $-\infty < t < \infty$

$$\boxed{X_2 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}}$$

second hidden vector (take $t = -2$).
