Name ______________________ **2270-1**

Introduction to Linear Algebra 2270-1
Midterm Exam 2 Spring 2004

Take-Home Problem 1 Due 23 March

Take-Home Problem 2 Due 26 March

In-class Exam Date: 30 March

Instructions. Take-home problems **1** and **2** below are to be submitted at class time on the date marked above. The in-class portion of the exam is 50 minutes, three problems, similar to those on the sample exam. Calculators, books, notes and computers are not allowed.

1. (Matrices, determinants and independence)

(a) Assume that $\det(EA) = \det(E)\det(A)$ holds for an elementary swap, multiply or combination matrix E and any square matrix A. Prove that elementary matrices $E_1, \ldots, E_k$ exist such that

$$\det(A) = \frac{\det(\mathbf{rref}(A))}{\det(E_1)\cdots\det(E_k)}.$$

(b) Prove that the column positions of leading ones in $\mathbf{rref}(A)$ identify columns of A which form a basis for $\mathbf{image}(A)$.

(c) Let the $n \times n$ matrix A be **nilpotent**, that is, $A^k = 0$ for some $k \geq 1$, but $A^{k-1} \neq 0$. Choose a vector $\mathbf{v}$ in $\mathcal{R}^n$ such that $A^{k-1}\mathbf{v} \neq \mathbf{0}$. Prove that $\mathbf{v}$, $A\mathbf{v}$, $\ldots$, $A^{k-1}\mathbf{v}$ are linearly independent.

(d) Let T be the linear transformation on $\mathcal{R}^3$ defined by mapping the columns of the identity respectively into three independent vectors $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$. Define $\mathbf{u}_1 = \mathbf{v}_1 + \mathbf{v}_3$, $\mathbf{u}_2 = \mathbf{v}_1 + \mathbf{v}_2$, $\mathbf{u}_3 = \mathbf{v}_2 + 2\mathbf{v}_3$. Verify that $\mathcal{B} = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is a basis for $\mathcal{R}^3$ and report the $\mathcal{B}$-matrix of T (Otto Bretscher page 139).

Please staple this page to the front of your submitted exam problem 1.

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Take-Home Problem 2 Due 26 March

In-class Exam Date: 30 March

2. (Kernel and similarity)

(a) Prove or disprove: $AB = I$ with A, B possibly non-square implies $\mathbf{ker}(A) = \{\mathbf{0}\}$.

(b) Prove or disprove: $\mathbf{ker}(\mathbf{rref}(BA)) = \mathbf{ker}(A)$, for all invertible matrices B.

(c) Find a matrix A of size 5×5 that is not similar to a diagonal matrix. Verify assertions.

(d) Let

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad T = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Prove or disprove: A is similar to the upper triangular matrix T.

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3. (Independence and bases)

(a) Let A be an $n \times m$ matrix. Find a condition on A such that independent vectors $\mathbf{v}_1, \ldots, \mathbf{v}_k$ are mapped by A into independent vectors $A\mathbf{v}_1, \ldots, A\mathbf{v}_k$. Prove assertions.

(b) Let V be the vector space of all polynomials $c_0 + c_1 x + c_2 x^2$ under function addition and scalar multiplication. Prove that $1 - x$, $2x$, $(x-1)^2$ form a basis of V.

Please staple this page to the front of your submitted exam problem 3.

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4. (Linear transformations)

(a) Let L be a line through the origin in $\mathcal{R}^2$ with unit direction $\mathbf{u}$. Let T be a reflection through L. Define T precisely. Display its representation matrix A, i.e., $T(\mathbf{x}) = A\mathbf{x}$.

(b) Let T be a linear transformation from $\mathcal{R}^n$ into $\mathcal{R}^m$. Given a basis $\mathbf{v}_1, \ldots, \mathbf{v}_n$ of $\mathcal{R}^n$, let A be the matrix whose columns are $T(\mathbf{v}_1), \ldots, T(\mathbf{v}_n)$. Prove that $T(\mathbf{x}) = A\mathbf{x}$.

Please staple this page to the front of your submitted exam problem 4.

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5. (Vector spaces)

(a) Show that the set of all 5×4 matrices A which have exactly one element equal to 1, and all other elements zero, form a basis for the vector space of all 5×4 matrices.

(b) Let W be the set of all functions defined on the real line, using the usual definitions of function addition and scalar multiplication. Let V be the set of all polynomials of degree less than 5 (e.g., $x^4 \in V$ but $x^5 \notin V$). Assume W is known to be a vector space. Prove that V is a subspace of W.

Please staple this page to the front of your submitted exam problem 5.