Name ________________________ **2270-1**

Introduction to Linear Algebra 2270-1
Midterm Exam 2 Fall 2003

Take-Home Exam Date: Friday, 17 October, 2003
In-class Exam Date: Tuesday, 21 October, 2003

Instructions. The four take-home problems below are to be submitted at class time at the date marked above. Answer checks are expected. If `maple` assist is used, then please attach the `maple` output.
The in-class portion of the exam is the last 15 minutes of class, one problem, of a type similar to either problem 3 or problem 4. Calculators are not allowed. Books and notes are not allowed.

1. **(Matrix facts)**
(a) Let A be a given matrix. Assume $\mathbf{rref}(A) = E_1 E_2 \cdots E_k A$ for some elementary matrices E_1, E_2, $\ldots$, E_k. Prove that if A is invertible, then A^{-1} is the product of elementary matrices.

(b) Suppose $A^2 = \mathbf{0}$ for some square matrix A. Prove that $I + 2A$ is invertible.

(c) Prove that a matrix with two equal rows cannot be invertible.

2. **(RGB and sunglasses)** Consider the equations

$$\begin{aligned} I &= \frac{1}{3}(R+G+B) \\ L &= R-G \\ S &= B-\frac{1}{2}(R+G). \end{aligned}$$

On page 90 of Otto Bretscher's text, these equations are discussed as representing the intensity I, long-wave signal L and short-wave signal S in terms of the amounts R, G, B of red, green and blue light. Submit parts (a), (b), (c), (d) from problem 76, page 90.

3. **(Kernel properties)**
(a) Prove or disprove: $\mathbf{ker}(\mathbf{rref}(A)) = \mathbf{ker}(A)$.

(b) Prove or disprove: $AB = I$ implies $\mathbf{ker}(B) = \{\mathbf{0}\}$.

(c) Find the best general values of c and d in the inequality $c \le \mathbf{dim}(\mathbf{ker}(A)) \le d$. The constants depend on the row and column dimensions of A.

(d) Prove that similar matrices A and B satisfy $\mathbf{nullity}(A) = \mathbf{nullity}(B)$.

4. **(Independence and bases)**
(a) Show that the set of all $m \times n$ matrices A which have exactly one element equal to 1, and all other elements zero, form a basis for the vector space of all $m \times n$ matrices.

(b) Let V be the vector space of all polynomials under function addition and scalar multiplication. Prove that 1, x, $\ldots$, x^n are independent in V.

(c) Let A be the matrix of a linear transformation T from R^n into R^n. Find a condition on A such that independent vectors v_1, $\ldots$, v_k are mapped by T into independent vectors $T(v_1)$, $\ldots$, $T(v_k)$. Prove assertions.

Please attach this exam or a copy to the front of your submitted exam on the due date. Kindly staple the left upper corner and write your name on all pages.