Ex. 1 Transform to a first order system

$$\begin{cases} 2x'' = -6x + 2y \\ y'' = 2x - 2y + 40 \sin 3t \end{cases}$$

Let $\begin{cases} x_1 = x \\ x_2 = x' \end{cases}$ and $\begin{cases} x_3 = y \\ x_4 = y' \end{cases}$ Then the system is

$$\begin{cases} x_1' = x_2 \\ x_2' = -3x_1 + x_3 \\ x_3' = x_4 \\ x_4' = 2x_1 - 2x_3 + 40 \sin 3t \end{cases}$$

← use $x_2' = x'' = -6x + 2y$

Ex. 2 Solve $\begin{cases} x' = x \\ y' = -y \end{cases}$ (an uncoupled system)

Each equation is a growth-decay equation in one variable. By recipe,

$$\begin{cases} x = x_0 e^{t} \\ y = y_0 e^{-t} \end{cases}$$

Ex. 3 Solve $\begin{cases} x' = x \\ y' = x + y \end{cases}$ (a triangular system)

Solve the decay-growth equation $x' = x$ by recipe:

$$\boxed{x = x_0 e^{t}}$$

Then substitute into $y' = x + y$ to get

$$y' = y + x_0 e^{t}.$$

Solve by linear methods:

$$y' - y = x_0 e^{t}$$

$$\frac{(e^{-t} y)'}{e^{-t}} = x_0 e^{t}$$

$$\boxed{y = x_0 t e^{t} + y_0 e^{t}}$$

Ex. 4. Solve $\begin{cases} x' = ax + by \\ y' = -bx + ay \end{cases}$ $(b > 0)$

answer: $\begin{cases} x = e^{at} (c_1 \cos bt + c_2 \sin bt) \\ y = e^{at} (-c_1 \sin bt + c_2 \cos bt) \end{cases}$

Let $X = x e^{-at}$, $Y = y e^{-at}$. Then

$$\begin{aligned} X' &= (x e^{-at})' \\ &= -a x e^{-at} + x' e^{-at} \\ &= -a X + (ax + by) e^{-at} \\ &= -a X + a X + b Y \\ &= b Y \end{aligned}$$

Similarly, $Y' = -b X$, so

$$\begin{cases} X' = b Y, \\ Y' = -b X. \end{cases}$$

Then $X'' = b Y' = -b^2 X$ and X satisfies $X'' + b X = 0$ with solution

$$X = c_1 \cos bt + c_2 \sin bt$$

$$\begin{aligned} x &= X e^{at} \\ &= (c_1 \cos bt + c_2 \sin bt) e^{at}, \end{aligned}$$

$$\begin{aligned} y &= Y e^{at} \\ &= \tfrac{1}{b} X' e^{at} \\ &= (-c_1 \sin bt + c_2 \cos bt) e^{at} \end{aligned}$$

Brine Mixing

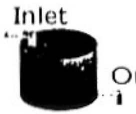

Figure 1. A brine tank with one inlet and one outlet.

A given tank contains brine, that is, water and salt. Input pipes supply other, possibly different brine mixtures at varying rates, while output pipes drain the tank. The problem is to determine the salt $x(t)$ in the tank at any time. The basic chemical law to be applied is the **mixture law**

$$\frac{dx}{dt} = \text{input rate} - \text{output rate}.$$

The law is applied under a simplifying assumption: *the concentration of salt in the brine is uniform throughout the fluid.* Stirring is one way to meet this requirement.

One Input and One Output.

Let the input be $a(t)$ liters per minute with concentration C_1 kilograms of salt per liter. Let the output empty $b(t)$ liters per minute. The tank is assumed to contain V_0 liters of brine at $t = 0$. The tank gains fluid at rate $a(t)$ and loses fluid at rate $b(t)$, therefore $V(t) = V_0 + \int_0^t [a(r) - b(r)]dr$ is the volume of brine in the tank at time t. The *mixture law* applies to obtain the model linear differential equation

$$\frac{dx}{dt} = C_1 a(t) - \frac{b(t)x(t)}{V(t)}.$$

Brine-Mixing One-tank Proof: The brine-mixing equation

$$x'(t) = C_1 a(t) - b(t)x(t)/V(t)$$

is justified for the one-tank model, by applying the *mixture law*

$$dx/dt = \text{input rate} - \text{output rate}$$

as follows.

$$\begin{aligned}
\text{input rate} &= \left(a(t)\frac{\text{liters}}{\text{minute}}\right)\left(C_1\frac{\text{kilograms}}{\text{liter}}\right) \\
&= C_1 a(t)\frac{\text{kilograms}}{\text{minute}}, \\
\text{output rate} &= \left(b(t)\frac{\text{liters}}{\text{minute}}\right)\left(\frac{x(t)}{V(t)}\frac{\text{kilograms}}{\text{liter}}\right) \\
&= \frac{b(t)x(t)}{V(t)}\frac{\text{kilograms}}{\text{minute}}.
\end{aligned}$$

Two-Tank Mixing. Two tanks A and B are assumed to contain A_0 and B_0 liters of brine at $t = 0$. Let the input for the first tank A be $a(t)$ liters per minute with concentration C_1 kilograms of salt per liter. Let tank A empty at $b(t)$ liters per minute into a second tank B, which itself empties at $c(t)$ liters per minute.

Let $x(t)$ be the number of kilograms of salt in tank A at time t. Similarly, $y(t)$ is the amount of salt in tank B. The *objective* is to find differential equations for the unknowns $x(t)$, $y(t)$.

Fluid loses and gains in each tank give rise to the brine volume formulas $V_A(t) = A_0 + \int_0^t [a(r) - b(r)]dr$ and $V_B(t) = B_0 + \int_0^t [b(r) - c(r)]dr$, respectively, for tanks A and B, at time t.

The *mixture law* applies to obtain the model linear differential equations

$$\frac{dx}{dt} = C_1 a(t) - \frac{b(t)x(t)}{V_A(t)}, \quad \frac{dy}{dt} = \frac{b(t)x(t)}{V_A(t)} - \frac{c(t)y(t)}{V_B(t)}.$$

The first equation was solved in the previous paragraph, hence there is an explicit formula for $x(t)$. Substitute this formula into the second equation, then solve for $y(t)$ (by the same method).

7.1-7
7.1-6 P393

write as a first order system

$$\begin{cases} x'' - 5x + 4y = 0 \\ y'' + 4x - 5y = 0 \end{cases}$$

Let $\begin{cases} x_1 = x \\ x_2 = x' \end{cases}$ and $\begin{cases} x_3 = y \\ x_4 = y' \end{cases}$, Then

$$\begin{cases} x_1' = x_2 \\ x_2' = 5x_1 - 4x_3 \\ x_3' = x_4 \\ x_4' = -4x_1 + 5x_3 \end{cases}$$

7.1-15
7.1-16 P394

Solve $\begin{cases} x' = 8y \\ y' = -2x \end{cases}$

$$x'' = 8y' = 8(-2x) = -16x$$

$$x'' + 4^2 x = 0$$

$$\boxed{x = c_1 \cos 4t + c_2 \sin 4t}$$

Recipe case 3

$$y = \frac{1}{8}x'$$

$$\boxed{y = \frac{1}{2}(-c_1 \sin 4t + c_2 \cos 4t)}$$

7.2-17
7.2-16 P408

$A = \begin{bmatrix} 4 & 1 \\ -2 & 1 \end{bmatrix}$, $\mathbf{x}' = A\mathbf{x}$, $x_1 = e^{3t}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$, $x_2 = e^{2t}$

(a) Verify x_1, x_2 are solutions. (b) Verify x_1, x_2 a independent.

(a) $x_1' = 3x_1$ $\quad$ $x_2' = 2x_2$

$$Ax_1 = e^{3t} A \begin{bmatrix} 1 \\ -1 \end{bmatrix} = e^{3t}\begin{bmatrix} +3 \\ -3 \end{bmatrix} = 3x_1 = x_1' \checkmark$$

$$Ax_2 = e^{2t} A \begin{bmatrix} 1 \\ -2 \end{bmatrix} = e^{2t}\begin{bmatrix} 2 \\ -4 \end{bmatrix} = 2x_2 = x_2'$$

(b) Wronskian (x_1, x_2)

$$= \begin{vmatrix} e^{3t} & e^{2t} \\ -e^{3t} & -2e^{2t} \end{vmatrix} = e^{3t}e^{2t}\begin{vmatrix} 1 & 1 \\ -1 & -2 \end{vmatrix} = -e^{5t} \neq 0$$

$\therefore$ independent by Theorem.

7.3-9 Solve $\begin{cases} x_1' = 2x_1 - 5x_2 \\ x_2' = 4x_1 - 2x_2 \\ x_1(0) = 2, \quad x_2(0) = 3 \end{cases}$

or $\mathbf{x}' = A\mathbf{x}$, $\mathbf{x}(0) = \mathbf{x}_0$ where $A = \begin{bmatrix} 2 & -5 \\ 4 & -2 \end{bmatrix}$, $\mathbf{x}_0 = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$.

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answer: $\mathbf{x} = c_1 \mathbf{x}_1 + c_2 \mathbf{x}_2$, $\mathbf{x}_1 = \mathrm{Re}\left(\begin{bmatrix} \frac{1}{2}+i \\ 1 \end{bmatrix} e^{4it}\right)$,
$\mathbf{x}_2 = \mathrm{Im}\left(\begin{bmatrix} \frac{1}{2}+i \\ 1 \end{bmatrix} e^{4it}\right)$. Given $\mathbf{x}_0 = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$, Then
$x_1 = 2\cos 4t - \frac{11}{4} \sin 4t$, $x_2 = 3\cos 4t + \frac{1}{2}\sin 4t$

Eigenanalysis $\det(A - \lambda I) = \lambda^2 + 16$, roots $\pm 4i$

$$(A - \lambda I) = \begin{bmatrix} 2-4i & -5 \\ 4 & -2-4i \end{bmatrix} \cong \begin{bmatrix} 1 & -\frac{1}{2}-i \\ 0 & 0 \end{bmatrix}$$

Eigenvector $= \begin{bmatrix} \frac{1}{2}+i \\ 1 \end{bmatrix} = \begin{bmatrix} 1/2 \\ 1 \end{bmatrix} + i \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$$\mathbf{x}_1 = \mathrm{Re}\left(\begin{bmatrix} \frac{1}{2}+i \\ 1 \end{bmatrix} e^{4it}\right)$$
$$= \mathrm{Re}\left[\left(\begin{bmatrix} 1/2 \\ 1 \end{bmatrix} + i \begin{bmatrix} 1 \\ 0 \end{bmatrix}\right)(\cos 4t + i \sin 4t)\right]$$
$$= \begin{bmatrix} 1/2 \\ 1 \end{bmatrix} \cos 4t - \begin{bmatrix} 1 \\ 0 \end{bmatrix} \sin 4t$$

$\mathbf{x}_2 = \mathrm{Im}$ (same)
$$= \begin{bmatrix} 1/2 \\ 1 \end{bmatrix} \sin 4t + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \cos 4t$$

To find the coefficients c_1, c_2 for $\mathbf{x}_0 = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$, solve
$c_1 \mathbf{x}_1(0) + c_2 \mathbf{x}_2(0) = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ for c_1, c_2. Then $c_1 = 4$, $c_2 = -1$.

7.3-19 Find the general solution by the eigenvalue metho[d]

$$x_1' = 4x_1 + x_2 + x_3$$
$$x_2' = x_1 + 4x_2 + x_3$$
$$x_3' = x_1 + x_2 + 4x_3$$

Let $A = \begin{bmatrix} 4 & 1 & 1 \\ 1 & 4 & 1 \\ 1 & 1 & 4 \end{bmatrix}$. Rephrase as $\mathbf{x}' = A\mathbf{x}$.

$\det(A - \lambda I) = -\lambda^3 + 12\lambda^2 - 45\lambda + 54$, roots $\lambda = 6, 3, 3$.

Eigenvectors are $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$.

$$\mathbf{x} = c_1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} e^{6t} + c_2 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} e^{3t} + c_3 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} e^{3t}$$

Required details:

- Evaluate $\det(A - \lambda I)$
- Factor the polynomial, find the roots
- Solve $(A - \lambda I)v = 0$ when $\lambda = 6$, get $v = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$
- Solve $(A - \lambda I)v = 0$ when $\lambda = 3$, get $v = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

Text answers for $\lambda = 3$ are $\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$, obtained from $t\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + s\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ by taking $t = -2, s = 1$ and $t = 0, s =$ · respectively.