Ch 6 Notes + Exercises

# Eigenanalysis

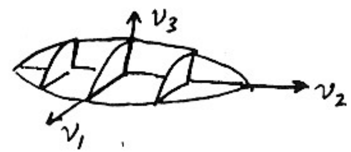
The subjects of eigenvalues and eigenvectors were created in order to *simplify* the equations which represent a physical problem. The equations can be algebraic equations, ordinary or partial differential equations, integral equations or infinite series expansions.

The simplifications are realized by finding a *special set of coordinates* for the problem, which transforms complexity out of the problem's equations. Some examples:

- Geometric problems
- Chemical kinetics
- Heat transfer
- Electrical networks
- Coupled mechanical systems

- In differential equations, eigenanalysis is used *to find easy-to-solve equations* related to the original problem by a change of variables.
- In geometry, eigenanalysis is used *to determine coordinate systems* which greatly simplify and standardize the corresponding algebraic equations for the geometric objects.
- In heat transfer, eigenanalysis is used *to determine a basis of functions*, by which the temperature function can be simply expressed. The ideas, due to Fourier and Dirichlet, led to *Fourier Integral* representations and to *generalized Fourier Series* expansions.

## A Geometry Problem: the Ellipsoid




an ellipsoid in $\mathbb{R}^3$

Eigenanalysis for an ellipsoid calculates the semiaxis directions $v_1, v_2, v_3$ and the semiaxis lengths $\ell_1, \ell_2, \ell_3$.

- In the coordinate system created by $v_1, v_2, v_3$, the ellipsoid equation takes the *standard simplified form*

(1) $$\frac{x^2}{\ell_1^2} + \frac{y^2}{\ell_2^2} + \frac{z^2}{\ell_3^2} = 1.$$

- The role of eigenanalysis is to calculate from a given ellipsoid equation, e.g.,

(2) $$x^2 + 2y^2 + 4z^2 + xz - xy = 10,$$

The *hidden coordinate system* $v_1, v_2, v_3$ and the *hidden values* $\frac{1}{\ell_1^2}, \frac{1}{\ell_2^2}, \frac{1}{\ell_3^2}$, which transforms (2) into (1).

- Briefly, eigenanalysis is a computational tool to find a *change of variables* to convert (2) into (1), hence removing the algebraic complexity from the ellipsoid's equation.

Let $A = \begin{bmatrix} 1 & -1 \\ -4 & 1 \end{bmatrix}$. Find the hidden vectors and hidden values of $A$.

## Hidden Values

$$\det(A - \lambda I) = \begin{vmatrix} 1-\lambda & -1 \\ -4 & 1-\lambda \end{vmatrix}$$ Hidden values $\lambda$ satisfy $\det(A - \lambda I) = 0$

$$= (1-\lambda)^2 - 4$$ Expand

$$= (\lambda+1)(\lambda-3)$$ Factor

$$\boxed{\lambda_1 = -1,\ \lambda_2 = 3}$$ Hidden values found.

## Hidden Vectors

$$A - \lambda_1 I = \begin{bmatrix} 1-\lambda_1 & -1 \\ -4 & 1-\lambda_1 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ -4 & 2 \end{bmatrix} \approx \begin{bmatrix} 2 & -1 \\ 0 & 0 \end{bmatrix}$$

RREF found; solution: $x = \frac{1}{2}t$, $y = t$, $-\infty < t < \infty$

$$\boxed{X_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}}$$

First hidden vector (take $t = 2$).

$$A - \lambda_2 I = \begin{bmatrix} 1-\lambda_2 & -1 \\ -4 & 1-\lambda_2 \end{bmatrix} = \begin{bmatrix} -2 & -1 \\ -4 & -2 \end{bmatrix} \approx \begin{bmatrix} 2 & 1 \\ 0 & 0 \end{bmatrix}$$

RREF found; solution: $x = -\frac{1}{2}t$, $y = t$, $-\infty < t < \infty$

$$\boxed{X_2 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}}$$

second hidden vector (take $t = -2$).

# Systems of Differential Equations

**Example.** Eigenanalysis finds the matrix
$$P = \begin{bmatrix} 1 & 1 \\ 2 & -2 \end{bmatrix}$$
and the hidden values $\lambda_1 = -1$, $\lambda_2 = 3$ such that the change of variables
$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$$
changes the coupled system

(1) $$\begin{cases} x_1' = x_1 - x_2 + 1, \\ x_2' = -4x_1 + x_2 + t, \end{cases}$$

into the easy-to-solve uncoupled system

(2) $$\begin{cases} z_1' = -z_1 + \frac{1}{2} - \frac{t}{4}, \\ z_2' = 3z_2 + \frac{1}{2} + \frac{t}{4}. \end{cases}$$

i) The matrix $P$ has columns $v_1, v_2$. These vectors replace the x-axis and y-axis directions in the new coordinate system.

ii) The columns $v_1, v_2$ of $P$ are called hidden vectors and $\lambda_1 = -1$, $\lambda_2 = 3$ are called hidden values, because the system (1) does not directly or easily expose the vectors $v_1, v_2$ or values $\lambda_1, \lambda_2$; but they are hidden in (1).

# Systems of Differential Equations

Mechanical systems and electrical networks are modeled by a differential system of the form

Coupled system

$$(1)\quad \begin{cases} y_1' = a_{11} y_1 + \cdots + a_{1n} y_n + f_1, \\ y_2' = a_{21} y_1 + \cdots + a_{2n} y_n + f_2, \\ \quad\vdots \\ y_n' = a_{n1} y_1 + \cdots + a_{nn} y_n + f_n, \end{cases}$$

where $A = [a_{ij}]$ is a constant matrix and $f_1(t), \ldots, f_n(t)$ are given functions of $t$.

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Under the right conditions, there is a change of variable $y = Pz$, with $P$ given and $z$ the new variable, which changes (1) into the uncoupled easy-to-solve system

Uncoupled system

$$(2)\quad \begin{cases} z_1' = \lambda_1 z_1 + g_1, \\ z_2' = \lambda_2 z_2 + g_2, \\ \quad\vdots \\ z_n' = \lambda_n z_n + g_n, \end{cases}$$

where $\lambda_1, \ldots, \lambda_n$ are constants and $g_1(t), \ldots, g_n(t)$ are functions of $t$.

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Eigenvectors and Eigenvalues

Eigenanalysis finds the change of coordinates

$$y = Pz$$

and the numbers $\lambda_1, \ldots, \lambda_n$ appearing in (2). When successful, eigenanalysis changes (1) into the easy-to-solve system (2).

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6.1-9, P362
6.1-7

$A = \begin{bmatrix} 10 & -8 \\ 6 & -4 \end{bmatrix}$

Find the eigenvalues and eigenvectors. Determine a basis for each eigenspace.

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Eigenvalues

$\det(A - \lambda I) = \begin{vmatrix} 10-\lambda & -8 \\ 6 & -4-\lambda \end{vmatrix}$

$= \lambda^2 - (10-4)\lambda + (-40+48)$ — Expand, Sarrus' rule

$= \lambda^2 - 6\lambda + 8$

$= (\lambda-2)(\lambda-4)$ — Factored

$\boxed{\lambda = 2,\ \lambda = 4}$ — Eigenvalues found

Eigenvectors and Bases for Eigenspaces.

$\begin{bmatrix} 8 & -8 \\ 6 & -6 \end{bmatrix}$ — $A - \lambda_1 I$ where $\lambda_1 = 2$

$\cong \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}$

$\cong \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$ — RREF found

$x = t,\ y = t$ — Solve it.

$\boxed{\text{Basis for } \lambda = 2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}}$ — Choose $t = 1$

$\begin{bmatrix} 6 & -8 \\ 6 & -8 \end{bmatrix}$ — $A - \lambda_2 I$ where $\lambda_2 = 4$

$\cong \begin{bmatrix} 3 & -4 \\ 0 & 0 \end{bmatrix}$

$\cong \begin{bmatrix} 1 & -4/3 \\ 0 & 0 \end{bmatrix}$ — RREF found

$x = 4t/3,\ y = t$ — Solve it

$\boxed{\text{Basis for } \lambda = 4 = \begin{bmatrix} 4 \\ 3 \end{bmatrix}}$ — Choose $t = 3$ to clear fractions

6.1-21, p362.

**6.1-19**

$$A = \begin{bmatrix} 4 & -3 & 1 \\ 2 & -1 & 1 \\ 0 & 0 & 2 \end{bmatrix}$$

Find eigenvalues, eigenvectors, bases for the eigenspaces.

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Eigenvalues

$$\det(A-\lambda I) = \begin{vmatrix} 4-\lambda & -3 & 1 \\ 2 & -1-\lambda & 1 \\ 0 & 0 & 2-\lambda \end{vmatrix}$$

$= (2-\lambda)\begin{vmatrix} 4-\lambda & -3 \\ 2 & -1-\lambda \end{vmatrix}$ Cofactor expansion on row 3.

$= (2-\lambda)[\lambda^2 - 3\lambda + 2]$ Expand, Sarrus' rule

$= (2-\lambda)(\lambda-2)(\lambda-1)$ Factor

$\boxed{\lambda = 1, \ \lambda = 2}$ Eigenvalues found

Eigenvectors and Bases

$\begin{bmatrix} 3 & -3 & 1 \\ 2 & -2 & 1 \\ 0 & 0 & 1 \end{bmatrix}$ $A - \lambda_1 I$ where $\lambda_1 = 1$

$\cong \begin{bmatrix} 3 & -3 & 0 \\ 2 & -2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ using row operations to find $\text{rref}(A-\lambda_1 I)$

$\cong \begin{bmatrix} 1 & -1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ ⋮

$\cong \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ RREF Found

$x = t, \ y = t, \ z = 0$ Solve it

$\boxed{\text{Basis} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \text{ For } \lambda = 1}$ Choose $t = 1$

— Continued —

6.1-19 continued

$\begin{bmatrix} 2 & -3 & 1 \\ 2 & -3 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ $A - \lambda_2$

$\cong \begin{bmatrix} 2 & -3 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$\cong \begin{bmatrix} 1 & -1.5 & .5 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ RREF

$x = 1.5t - 0.5s, \ y = t, \ z = s$ Solve

$\boxed{\text{Basis} = \begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}}$ choos Then Get vecto

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Check:

$\lambda = 1$: $\begin{bmatrix} 3 & -3 & 1 \\ 2 & -2 & 1 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 3-3 \\ 2-2 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ ✓

$\lambda = 2$: $\begin{bmatrix} 2 & -3 & 1 \\ 2 & -3 & 1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 6-6 \\ 6-6 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ ✓

$\begin{bmatrix} 2 & -3 & 1 \\ 2 & -3 & 1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix} = \begin{bmatrix} 2-2 \\ 2-2 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ ✓

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6.1-29, P363
6.1-31

$A = \begin{bmatrix} 0 & 24 \\ -6 & 0 \end{bmatrix}$ Find complex eigenvalues and eigenvectors.

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$\det(A - \lambda I) = 0$ Solve for eigenvalues

$\lambda^2 + (6)(24) = 0$

$\boxed{\lambda = \pm 12i}$ complex roots

Eigenvectors and Bases

$A - (12i)I$

$\begin{bmatrix} -12i & 24 \\ -6 & -12i \end{bmatrix}$

$\cong \begin{bmatrix} i & -2 \\ 1 & 2i \end{bmatrix}$

$\cong \begin{bmatrix} 1 & 2i \\ i & -2 \end{bmatrix}$

$\cong \begin{bmatrix} 1 & 2i \\ 0 & 0 \end{bmatrix}$ RREF Found

Solve it.

$x = -2it,\ y = t$

Choose $t = 1$

$\boxed{\text{Basis} = \begin{bmatrix} -2i \\ 1 \end{bmatrix} \text{ For } \lambda = 12i}$

$A \begin{bmatrix} 2i \\ 1 \end{bmatrix} = -12i \begin{bmatrix} 2i \\ 1 \end{bmatrix}$ Take conjugates, since A is real (theory applied)

$[A - (-12i)I] \begin{bmatrix} 2i \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ algebra

$\boxed{\text{Basis} = \begin{bmatrix} 2i \\ 1 \end{bmatrix} \text{ For } \lambda = -12i}$

Check:

$A \begin{bmatrix} -2i \\ 1 \end{bmatrix} = \begin{bmatrix} 0 & 24 \\ -6 & 0 \end{bmatrix} \begin{bmatrix} -2i \\ 1 \end{bmatrix} = \begin{bmatrix} 24 \\ 12i \end{bmatrix} = 12i \begin{bmatrix} -2i \\ 1 \end{bmatrix}$ ✓

$A \begin{bmatrix} 2i \\ 1 \end{bmatrix} = \begin{bmatrix} 0 & 24 \\ -6 & 0 \end{bmatrix} \begin{bmatrix} 2i \\ 1 \end{bmatrix} = \begin{bmatrix} 24 \\ -12i \end{bmatrix} = -12i \begin{bmatrix} 2i \\ 1 \end{bmatrix}$ ✓

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6.2-11, P371
6.2-9

$A = \begin{bmatrix} -1 & 4 \\ -1 & 3 \end{bmatrix}$ Determine if diagonaliz and if so find P and D s.t. $P^{-1}AP = D$.

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$\det(A - \lambda I) = (-1-\lambda)(3-\lambda) + 4$

$= -3 - 3\lambda + \lambda + \lambda^2 + 4$

$= \lambda^2 - 2\lambda + 1$

$= (\lambda - 1)^2$

$\boxed{\lambda = 1}$, double root Eigenvalues found

$A - (1)I = \begin{bmatrix} -2 & 4 \\ -1 & 2 \end{bmatrix}$

$\cong \begin{bmatrix} 1 & -2 \\ 1 & -2 \end{bmatrix}$

$\cong \begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix}$ RREF found

Solve it.

$x = 2t,\ y = t$

$\boxed{\text{Not diagonalizable}}$ not enough independent eigenvectors.

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Example Repeat for $A = \begin{bmatrix} 2 & 5 \\ 0 & -3 \end{bmatrix}$.

$\det(A - \lambda I) = (2-\lambda)(-3-\lambda)$

$\boxed{\lambda = 2,\ \lambda = -3}$ Eigenvalues found

$A - (2)I = \begin{bmatrix} 0 & 5 \\ 0 & -5 \end{bmatrix}$

$\cong \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$

Basis $= \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$A - (-3)I = \begin{bmatrix} 5 & 5 \\ 0 & 0 \end{bmatrix}$

$\cong \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$

Basis $= \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

$\boxed{\text{Diagonalizable}}$ because there are 2 independent eigenvectors.

6.2-17, P371
6.2-15

$A = \begin{bmatrix} 3 & -3 & 1 \\ 2 & -2 & 1 \\ 0 & 0 & 1 \end{bmatrix}$   Find P and D, if possible, such that $P^{-1}AP = D$.

**Eigenvalues**

- $\det(A - \lambda I) = (1-\lambda)\begin{vmatrix} 3-\lambda & -3 \\ 2 & -2-\lambda \end{vmatrix}$   Cofactor expansion, row 3

$= -\lambda(\lambda - 1)^2$   Sarrus' rule; then factor

$\boxed{\lambda = 0, \lambda = 1}$   Eigenvalues found.

**Eigenvectors**

- $A - (0)I = \begin{bmatrix} 3 & -3 & 1 \\ 2 & -2 & 1 \\ 0 & 0 & 1 \end{bmatrix}$   Apply row operations to find ...

$\cong \begin{bmatrix} 3 & -3 & 0 \\ 2 & -2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$\cong \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$   The rref.

$x = t,\ y = t,\ z = 0$   solve it.

$\boxed{\text{Basis} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \text{ For } \lambda = 0}$   Take $t = 1$.

- $A - (1)I = \begin{bmatrix} 2 & -3 & 1 \\ 2 & -3 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

$\cong \begin{bmatrix} 1 & -3/2 & 1/2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$   RREF found.

$x = \frac{3}{2}t - \frac{1}{2}s,\ y = t,\ z = s$   Solve it.

$\boxed{\text{Basis} = \begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix} \text{ For } \lambda = 1}$   take $t = 2, s = 0$; then $t = 0, s = 2$.

**matrices P and D**

$$\boxed{P = \begin{bmatrix} 1 & 3 & -1 \\ 1 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} \quad D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}$$

The columns of P are the 3 eigenvectors of A.

Diagonal entries of D are 0, 1, 1, the eigenvalues of A.

Answer check: Book order of eigenvalues is different, and bases are different. But hand checking shows the above answers also correct.

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6.2-25

$A = \begin{bmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$   Find P and D s.t. $P^{-1}AP = D$.

Eigenvalues: $\lambda = 1,\ \lambda = -1$ (double roots)

Eigenvectors: $\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$ for $\lambda = 1$

$\begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$ for $\lambda = -1$

$P = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}.$

Check: $AP = \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}, \quad PD = \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$

$\therefore AP = PD$ ✓

**Details Required**

- Expand $\det(A - \lambda I)$ to get $(1-\lambda)^2(-1-\lambda)^2$. Use determinant results about upper triangular matric
- Find $\text{rref}(A - (1)I)$. Solve to find a basis of two vectors for eigenspace $\lambda = 1$.
- Find $\text{rref}(A - (-1)I)$. The eigenspace for $\lambda = -1$ has two elements.
- Assemble the eigenvalues into the diagonal elements of D. Copy the corresponding eigenvectors into the colu of P.
- check $P^{-1}AP = D$ by expanding AP and PD, show instead AP equals PD.