p 84 Separate variables and use partial fractions to solve the initial value problem

$$\frac{dx}{dt} = 3x(x-5), \qquad x(0) = 2.$$

$\dfrac{x'}{x(x-5)} = 3$	separated form
$\dfrac{Ax'}{x} + \dfrac{Bx'}{x-5} = 3$	partial fractions applied. constants A, B found later.
$A\ln\lvert x\rvert + B\ln\lvert x-5\rvert = 3t + c$	Method of quadrature
$A\ln 2 + B\ln 3 = c$	Evaluate when $t=0$, $x=2$ to find c.
$A\ln\dfrac{\lvert x\rvert}{2} + B\ln\dfrac{\lvert x-5\rvert}{3} = 3t$	Collect terms, using $\ln u - \ln v = \ln(u/v)$
$A = -1/5$, $B = 1/5$ in relation $\dfrac{1}{x(x-5)} = \dfrac{A}{x} + \dfrac{B}{x-5}$	Heaviside's coverup method applied
$-\ln\dfrac{\lvert x\rvert}{2} + \ln\dfrac{\lvert x-5\rvert}{3} = 15t$	Let $A = -1/5$, $B = 1/5$ and multiply by 5.
$\ln\dfrac{2}{\lvert x\rvert}\dfrac{\lvert x-5\rvert}{3} = 15t$	use log rule $\ln u - \ln v = \ln(u/v)$
$\dfrac{2}{3}\,\dfrac{5-x}{x} = e^{15t}$	Take logs, drop absolute values near $x=2$.
$x = \dfrac{5}{1 + \frac{3}{2}e^{15t}}$	solve for x.

$\left|\dfrac{x-5}{x}\right| = \dfrac{5-x}{x}$ near $x = 2$
[used $x = 2$ at $t = 0$]

#12 p 84 A rabbit population satisfies $P' = aP - bP^2$, $P(0) = P$. Given $P_0 = 120$, $aP_0 = 8$ births per month, $bP_0^2 = 6$ deaths per month, Then find the number T of months when $P(T)$ is 95% of the limiting population $M = c$

The given constants are $P_0 = 120$, $a = 1/15$, $b = 6/120^2$, $M = 160$. Then page 77 gives

$$P(T) = \frac{MP_0}{P_0 + (M - P_0)e^{-aT}}$$

and $P(T) = \frac{95}{100}M$ implies the exponential relation

$$\frac{95}{100}M = \frac{P_0 M}{P_0 + (M-P_0)e^{-aT}}.$$

Cancel M, clear fractions, rearrange terms to get

$$e^{-aT} = \frac{5}{95}\,\frac{P_0}{M - P_0} = \frac{5}{95}\cdot\frac{120}{40} = \frac{15}{95}$$

Taking logs gives

$$T = 15\ln\left(\frac{95}{15}\right).$$

#13, p84: $T = \frac{80}{3}\ln\left(\frac{21}{4}\right) \cong 44{,}22$

#4
#10 Find The critical points, classify as stable or unstable and construct The phase diagram. Sketch Typical solution curves. Finally, solve The DE explicitly.

2.2-4, p 93. $\frac{dx}{dt} = 3x - x^2$

$f(x) = 3x - x^2$ RHS of The DE.

$0 = x(3-x)$ Find critical points.

$x = 0,\ x = 3$

$f(x)<0$ $f(x)>0$ $f(x)<0$ phase diagram (see page 90)

$x=0$ unstable $x=3$ stable

$x(t) = \dfrac{3x_0}{x_0 + (3-x_0)e^{-3t}}$ See p 91, eqtn (10)

Funnel along $x=3$
Spout along $x=0$ see figure 2.2.4, p 90

2.2-10, p 93. $\frac{dx}{dt} = 7x - x^2 - 10$ (Harvesting)

$f(x) = 7x - x^2 - 10$ RHS of The DE

$0 = 7x - x^2 - 10$

$0 = -(x-5)(x-2)$

$x = 2,\ x = 5$ critical points found

$f(x)<0$ $f(x)>0$ $f(x)<0$ phase diagram

$x=2$ unstable $x=5$ stable

Funnel along $x = 5$
Spout along $x = 2$ see figure 2.2.8, p 92

Sketches duplicate The figures on p 90 and p 92. Submitted solutions should contain copies of These figures.

p104
#9 Motorboat problem

The equation has The form $v' + pv = q$, which is linear. Solving it gives

$$v = 50(1 - e^{-0.1t}).$$

The limit at ∞ is 50 ft/sec or 34 mph. In your solution:

- Solve $1000v' = 5000 - 100v$, $v(0)=0$.
- Take The limit, watch units.

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p105
#27 Projectile problem

The suggestion in The problem can be replaced

Show This step → by "multiply The DE by y' and apply The method of quadrature to variable $v = y'$."

The maximum height is found by setting $y'=0$ (or $v=0$). This is The equation

$$0 = R(R+y)v_0^2 - 2GMy,$$

because a fraction $\frac{A}{B}$ is zero only in case $A=0$.

To solve for y is routine, but to evaluate you must locate values for R, G, M and use The given $v_0 = 1$ km/sec.

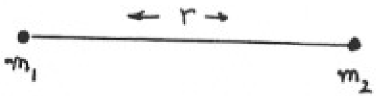
answer: 51.427 kilometers.

$G = 6.6726\times 10$
$M = 5.975\times 10$
$R = 6.378\times 10$

Escape Velocity of the Earth

Physics Background

Newton's universal law of gravitation $\qquad \text{Force} = \dfrac{m_1 m_2 G}{r^2}$

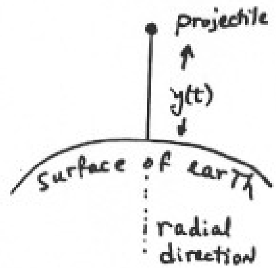


Acceleration g due to gravity $\qquad g = \dfrac{G m_2}{R^2}$

m_2 = Mass of the earth
R = Mean radius of the earth
G = universal Gravitation constant

Differential Equation Theory

- Ignore air resistance
- Let $y(t)$ = projectile-to-surface distance at time t.
- Let m_1 = mass of projectile, m_2 = mass of the earth.



R = mean radius of the earth

DE: $\quad m_1 y'' = -\dfrac{G m_1 m_2}{(R+y)^2}$

IC: $\quad y(0) = 0,\ y'(0) = v_0$

Escape Velocity of the Earth

Assumptions

- Velocity v_0 causes the projectile to exit the field of the earth and never return.
- v_0 is minimal.

Math formulation

- $\lim_{t \to \infty} y(t) = \infty$
- v_0 = minimum over all possible v_0

Derivation of $v_0 = \sqrt{2gR}$

$y''y' = -\dfrac{G m_2}{R^2} \dfrac{R^2 y'}{(R+y(t))^2}$ $\qquad$ mult DE by y' and cancel m_1

$\frac{1}{2}[(y')^2]' = -gR^2 \dfrac{y'}{(R+y)^2}$ $\qquad$ simplify constants write LHS as derivat

$\frac{1}{2}(y')^2 = -gR^2\left(\dfrac{-1}{R+y}\right) + c$ $\qquad$ method of quadratur applied.

$\frac{1}{2} v_0^2 = gR + c$ $\qquad$ Use $y=0$, $y'=v_0$ at $t=0$. Found c.

$0 \le c$ $\qquad$ Use $(y')^2 \ge 0$ and $\lim_{t\to\infty} y(t) = \infty$.

$c \le v_0^2 - 2gR$ $\qquad$ substitute for C.

$v_0 = \sqrt{2gR}$ $\qquad$ Minimum v_0 found.

Brine Mixing

Figure 1. A brine tank with one inlet and one outlet.

A given tank contains brine, that is, water and salt. In put pipes supply other, possibly different brine mixture at varying rates, while output pipes drain the tank. The problem is to determine the salt $x(t)$ in the tank at any time. The basic chemical law to be applied is the **mixture law**

$$\frac{dx}{dt} = \text{input rate} - \text{output rate}.$$

The law is applied under a simplifying assumption: *the concentration of salt in the brine is uniform throughou the fluid*. Stirring is one way to meet this requirement.

One Input and One Output.

Let the input be $a(t)$ liters per minute with concentration C_1 kilo grams of salt per liter. Let the output empty $b(t)$ liter per minute. The tank is assumed to contain V_0 liters o brine at $t = 0$. The tank gains fluid at rate $a(t)$ and lose fluid at rate $b(t)$, therefore $V(t) = V_0 + \int_0^t [a(r) - b(r)]dr$ i the volume of brine in the tank at time t. The *mixture law* applies to obtain the model linear differential equatio

$$\frac{dx}{dt} = C_1 a(t) - \frac{b(t)x(t)}{}.$$