Taking log gives:

\[
\frac{5b}{5} = \frac{40}{5} \Rightarrow 5b = 40 - 5p = 0
\]

Solve for \( x \).

The log rule gives:

\[
\log_{10} x = \log_{10} (x/5)
\]

and multiplying by 5:

\[
5 \log_{10} x = 5 \log_{10} (x/5)
\]

Let \( A = -15 \), \( B = 1/5 \)

Heaviside Covering

\[
\text{Evaluated when } t = 0
\]

Simplified form:

\[
x(t) = 3\text{x}(t-5) + \text{t}
\]

Given \( a = 10 \), \( b = 1/5 \)

The given constants are \( p = 120 \), \( a = 1/5 \).

\[
(x - 5) + \frac{3}{5} = \frac{3}{x - 5}
\]

\[
x = \frac{3}{5} \Rightarrow x = 1.5
\]

\[
x = 1/5, x = 1/5 \text{ in retention}
\]

\[
\text{At } x = 1/5, b = 1/5 \text{ in retention}
\]

\[
\text{At } x = 1/5, b = 3 = c
\]

\[
x(t) = 3\text{x}(t-5) + \text{t}
\]
Find the critical points. Classify as stable or unstable and construct the phase diagram. Sketch typical solution curves. Finally, solve the DE implicitly.

\[ \frac{dx}{dt} = 3x - x^2 \]

\[ f(x) = 3x - x^2 \]

\[ o = x(2-x) \]

\[ x = 0, \ x = 3 \]

\[ f(x) = 0 \]

\[ f(x) > 0 \quad f(x) < 0 \]

Phase diagram (See page 90)

Motorboat problem

The equation has the form \( \nu' + \rho \nu = 0 \), which is linear. Solving it gives

\[ \nu = 50(1 - e^{-0.1t}) \]

The limit at \( \infty \) is \( 50 \) ft/sec or 34 mph. In your solution:

- Solve \( 1000 \nu' = 5000 - 100 \nu, \ \nu(0) = 0. \)
- Take the limit, watch units.

Projectile problem

The suggestion in the problem can be replaced by "multiply the DE by \( \dot{y} \) and apply the method of quadrature to variable \( \nu = \dot{y} \)."

The maximum height is found by setting \( y' = 0 \) (or \( \nu = 0 \)). This is the equation

\[ 0 = R(R+y)\nu_0^2 - 2GMy \]

because a fraction \( \frac{A}{B} \) is zero only in case \( A = 0 \).

To solve for \( y \) is routine, but to evaluate you must locate values for \( R, G, M \) and use the given \( \nu_0 = 1 \) km/sec.

Answer: 51.427 kilometers.
Escape Velocity of the Earth

Physics Background

Newton's Universal Law of Gravitation

\[ \text{Force} = \frac{m_1 m_2 G}{r^2} \]

Acceleration \( g \) due to gravity

\[ g = \frac{G m_2}{R^2} \]

- \( m_2 \) = Mass of the Earth
- \( R \) = Mean radius of the Earth
- \( G \) = Universal Gravitation Constant

Differential Equation Theory

- Ignore air resistance
- Let \( y(t) \) = projectile-to-surface distance at time \( t \)
- Let \( m_1 \) = mass of projectile, \( m_2 \) = mass of the Earth

DE:

\[ m_1 \ddot{y} = -\frac{G m_1 m_2}{(R+y)^2} \]

IC:

\[ y(0) = 0, \quad y'(0) = v_0 \]

Assumptions

- Velocity \( v_0 \) causes the projectile to exit the field of the Earth and never return.
- \( v_0 \) is minimal.

Math formulation

- \( \lim_{t \to \infty} y(t) = \infty \)
- \( v_0 \) = Minimum over all possible \( v_0 \)

Derivation of \( v_0 = \sqrt{2gR} \)

\[ \frac{1}{2} \left( \frac{y'}{R+y} \right)^2 = -\frac{G m_2}{R} \left( \frac{y'}{R+y} \right) + C \]

\[ \frac{1}{2} v_0^2 = gR + C \]

0 \leq C \leq v_0^2 - 2gR

\[ v_0 = \sqrt{2gR} \]

Minimum \( v_0 \) found.
Brine Mixing

A given tank contains brine, that is, water and salt. Inlet pipes supply other, possibly different brine mixtures at varying rates, while output pipes drain the tank. The problem is to determine the salt $x(t)$ in the tank at any time. The basic chemical law to be applied is the mixture law

$$\frac{dx}{dt} = \text{input rate} - \text{output rate}.$$ 

The law is applied under a simplifying assumption: the concentration of salt in the brine is uniform throughout the fluid. Stirring is one way to meet this requirement.

One Input and One Output. Let the input be $a(t)$ liters per minute with concentration $C_1$ kilograms of salt per liter. Let the output empty $b(t)$ liters per minute. The tank is assumed to contain $V_0$ liters of brine at $t = 0$. The tank gains fluid at rate $a(t)$ and loses fluid at rate $b(t)$, therefore $V(t) = V_0 + \int_0^t [a(r) - b(r)]dr$ is the volume of brine in the tank at time $t$. The mixture law applies to obtain the model linear differential equation

$$\frac{dx}{dt} = C_1 a(t) - \frac{b(t)x(t)}{V(t)}.$$