PHQ #1. Solve \( \frac{dy}{dx} + 2xy = 0 \). Find the implicit and explicit solutions.

\[
\begin{align*}
\frac{dy}{y} &= -2x \, dx \\
\int \frac{dy}{y} &= \int -2x \, dx \\
\ln|y| &= -x^2 + c \\
|y| &= e^{-x^2 + c} \\
y &= e^{-x^2} + c \\
y &= y_0 e^{-x^2} \\
\text{check:} \\
LHS &= y' \\
&= (y_0 e^{-x^2})' \\
&= y_0 e^{-x^2} (-2x) \\
&= y_0 (-2x) \\
&= -2xy \\
&= \text{RHS}
\end{align*}
\]

PHQ #5. Solve \( 2\sqrt{x} \frac{dy}{dx} = \sqrt{1-y^2} \). Find the implicit, explicit, and equilibrium solutions.

Equilibrium solutions:

\[
\begin{align*}
0 &= \text{RHS of DE} \\
0 &= \sqrt{1-y^2} \\
0 &= (1-y)(1+y) \\
y &= 1 \text{ or } y = -1
\end{align*}
\]

Implicit non-equil sol:

\[
\frac{y'}{\sqrt{1-y^2}} = \frac{1}{2\sqrt{x}}
\]

Move \( \frac{1}{\sqrt{1-y^2}} \) to right side.

\[
(1-y)^{-\frac{1}{2}} y' dx = \frac{1}{2} \int x^{-\frac{1}{2}} dx
\]

\[
\sin^{-1}(x) = x + \frac{1}{2} + C
\]

Explicit sol.

\[
y(x) = \sin \left( x + \frac{1}{2} + C \right)
\]

Apply sine to both sides of implicit sol above.

Note: \( y = 1 \) and \( y = -1 \) are not included in the formula.

Check (checks with book, but explain why):

\[
LHS = 2x^\frac{1}{2} y' \\
= 2x^\frac{1}{2} \cos \left( x + \frac{1}{2} \right) \left( \frac{1}{2} x^{-\frac{1}{2}} \right) \\
= \cos \left( x + \frac{1}{2} \right)
\]

\[
RHS = \sqrt{1-y^2} \\
= \sqrt{1-\sin^2 (x + \frac{1}{2})} \\
= \pm \cos (x + \frac{1}{2})
\]
*11. \( y' = xy^3 \)

The separated form \( y'y^{-3} = x \) is easy enough to solve by quadrature, but be aware that \( y = 0 \) is an equilibrium, maybe not present in the derived formula.

*17. \( y' = 1 + x + y + xy \)

\[ (1+x)(1+y) \]

The separated form \( \frac{y'}{1+y} = 1 + x \) assumes \( y \neq -1 \). But \( y = -1 \) is an equilibrium.

*19. \( \frac{dy}{dx} = ye^x \)

The equilibrium \( y = 0 \) can be included in the explicit solution form, obtained by quadrature of the separated form \( \frac{ye^x}{y} = e^x \).

*27. \( \frac{dy}{dx} = 6e^{2x-y} \)

There are no equilibrium solutions. No separated form \( e^{y}y' = 6e^{2x} \) is found by applying the DE by \( e^y \).

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4.27 p40

Find equilibria, implicit and explicit solutions for \( y' = 6e^{2x-y}, y(0) = 0 \).

**Equilibria**

Solve \( 6e^{2x-y} = 0 \); no solutions; \( y = \) constant, so no equilibria.

**Implicit Solution**

\[ e^y = 6e^{2x} \]

\[ e^y = 3e^{2x} + c \]

\[ e^y = 3e^{2x} - 2 \]

Implicit sol.

**Explicit Solution**

Solve for \( y \) as a function of \( x \) in the implicit form.

\[ e^y = 3e^{2x} - 2 \]

\[ y = \ln|3e^{2x} - 2| \]

Given above

Take logs

**Answer Check**

\[ e^y = 3e^{2x} - 2 \]

\[ e^y' = 3(2xe^{2x}) \]

\[ y' = 6e^{2x}/e^y \]

\[ = 6e^{2x-y} \]

\[ y(0) = \ln|3e^{0} - 2| \]

\[ = \ln 1 \]

\[ = 0 \]

IC Checks also.
Solve the linear problem
\[ y' = y + e^x \]

**Solution**

**Standard Form**
\[ y' + (-1)y = e^x \]

**Factor**
\[ P = \int p(x) \, dx = \int -1 \, dx = -x \]
\[ e^P = e^{-x} \]

**Quadrature Form**
\[ y' + (-1)yg = e^x \]
\[ \frac{(e^P y')'}{e^P} = e^x \]
\[ \frac{e^x}{e^{-x}} = e^{2x} \]
\[ (e^{-x}y)' = 1 \]

**Method of Quadrature**
\[ \int (e^{2x}y)' \, dx = \int dx \]
\[ e^x y = x + C \]
\[ y = Ce^x + xe^x \]

**Ans.** Check next...

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**Problem 532**
\[ y' + 3y = 2xe^{-3x} \]

**Solution by \( \mu \)-factorization method**

**Standard Form**
\[ y' + p(x)y = s(x) \]

**\( \mu \)**
\[ P = \int \mu(x) \, dx = \int -3x \, dx = -\frac{3}{2}x^2 \]
\[ e^P = e^{-\frac{3}{2}x^2} \]

**Finding Quadrature Form**
\[ y' + (3)y = 2xe^{-3x} \]
\[ \frac{(e^P y')'}{e^P} = 2xe^{-3x} \]
\[ \frac{(e^{3x}y)'}{e^{3x}} = 2xe^{-3x} \]
\[ (e^{3x}y)' = 2xe^{-3x} \]

**Apply Method of Quadrature**
\[ s \int (e^{3x}y)' \, dx = \int 2xe^{-3x} \, dx \]
\[ e^{3x}y = x^2 + C \]
\[ y = (x^2 + C)e^{-3x} \]

**Report ans. and check**

**Ans.** Checks with textbook...
Solve by factorization method.

\[ y' + \left( \frac{2}{x} \right) y = 3 \]

\[ P = S \left( \frac{2}{x} \right) dx \]
\[ = 2 \ln x \]
\[ = \ln x^2 \]
\[ e^P = e^{\ln x^2} \]
\[ = x^2 \]

Find Quadrature Form

\[ y' + \left( \frac{2}{x} \right) y = 3 \]
\[ \frac{(e^P y)'}{e^P} = 3 \]
\[ \frac{(x^2 y)'}{x^2} = 3 \]
\[ (x^2 y)' = 3x^2 \]

Quadrature form found

\[ \int (x^2 y)' dx = \int 3x^2 dx \]
\[ x^2 y = x^3 + C \]
\[ y = x + \frac{C}{x^2} \]

Divide. Solution candidate found.

Replace \( y' + \frac{2}{x} y \) by \( \frac{(e^P y)'}{e^P} \)

Simplify \( e^P \) found

\[ \text{standard form } y' + py = q \]

Primitive \( P = \int p \, dx \)

\[ \text{Solution } y = x + \frac{C}{x^2} \]

Warning. Simple models like
\[ \begin{cases} y' = 3y^{2/3} \\ y(0) = 0 \end{cases} \]

Can be fed into computer algebra systems and numeric software packages. This problem has no "answer" but in all cases the computer gives an answer and no complaint.

The lesson is that computers are stupid, and engineers have to supply the logic and intuition, in order to make use of them.