Background

$\ln e^x = x, \quad e^{\ln y} = y$	In words, the exponential and the logarithm are inverses. The domains are $-\infty < x < \infty$, $0 < y < \infty$.
$e^0 = 1, \quad \ln(1) = 0$	Special values, usually memorized.
$e^{a+b} = e^a e^b$	In words, the exponential of a sum of terms is the product of the exponentials of the terms.
$(e^a)^b = e^{ab}$	Negatives are allowed, e.g., $(e^a)^{-1} = e^{-a}$.
$\left(e^{u(t)}\right)' = u'(t)e^{u(t)}$	The *chain rule* of calculus implies this formula from the identity $(e^x)' = e^x$.
$\ln AB = \ln A + \ln B$	In words, the logarithm of a product of factors is the sum of the logarithms of the factors.
$B \ln(A) = \ln\left(A^B\right)$	Negatives are allowed, e.g., $-\ln A = \ln \dfrac{1}{A}$.
$(\ln\|u(t)\|)' = \dfrac{u'(t)}{u(t)}$	The identity $(\ln(x))' = 1/x$ implies this general version by the *chain rule*.

PROBLEM 2: $\boxed{\text{Pg 17 \#2}}$

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Find a function $y = f(x)$ which satisfies the given differential equation $\frac{dy}{dx} = (x-2)^2$ and initial condition $y(2)=1$.

$y'(x) = (x-2)^2$ — given initial equation;

$y'(x)\,dx = (x-2)^2\,dx$ — apply the method of quadrature

$\int y'(x)\,dx = \int (x-2)^2\,dx$

$y(x) = \frac{(x-2)^3}{3} + c$

$1 = \frac{(2-2)^3}{3} + c$ — use $y(2)=1$

$c = 1$

$y(x) = \frac{(x-2)^3}{3} + 1$ — candidate solution;

Check:

$LHS = y'(x)$

$= \left[\frac{(x-2)^3}{3} + 1\right]'$

$= (x-2)^2 + 0$

$= RHS$ — checks with initial differential equation

$LHS = y(2)$

$= \left[\frac{(x-2)^3}{3} + 1\right]\Big|_{x=2}$

$= 0 + 1$

$= RHS$ — checks with initial condition $y(2)=1$.

1 Example (Decay Law Derivation) Derive the decay law $\frac{dA}{dt} = kA(t)$ from the sentence

> *Radioactive material decays at a rate proportional to the amount present.*

Solution: The sentence is first dissected into English phrases **1** to **4**.

1: *Radioactive material*	The phrase causes the invention of a symbol A for the amount present at time t.
2: *decays at a rate*	It means A undergoes decay. Then A changes. Calculus conventions imply the rate of change is dA/dt.
3: *proportional to*	Literally, it means *equal to a constant multiple of.* Let k be the proportionality constant.
4: *the amount present*	The amount of radioactive material present is $A(t)$.

Solution: *Continued ...*

The four phrases are translated into mathematical notation as follows.

Phrases 1 and 2	Symbol dA/dt.
Phrase 3	Equal sign '$=$' and a constant k.
Phrase 4	Symbol $A(t)$.

Let $A(t)$ be the amount present at time t. The translation is $\frac{dA}{dt} = kA(t)$.

To define the grid points, let $y = -1$ to 2 in increments of 0.3 to make 11 horizontal lines. The intersections account for a total of 109 grid points. It is possible to graph rapidly the 21 parabolas, because they are translates of $Y^2 = X$. The replacement segments, identical on each parabola, are also sketched rapidly. A computer graphic is shown in Figure 3 which closely resembles a hand-made graphic. Compare it to the graphic for the uniform grid method, Figure 2, page 72.

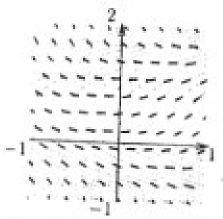

Figure 3. Direction field by the isocline grid method for $y' = x + y(1-y)$ on $-1 \le x \le 1$, $-1 \le y \le 2$.

The maple code that produced Figure 3 is included below as evidence that a hand computation is conceptually and mechanically easier. Sometimes a computer algebra system helps, especially to solve the equation $f(x, y) = M_i$ for y.

```
a:=-1:b:=1:n:=21:c:=-1:d:=2:m:=11:
H:=0.1:F:=-3:G:=1:f:=(x,y)->x+y*(1-y):
Slope:=x->F+1/4+(G-F)*(x-1)/(n-1):
Y:=x->c+(d-c)*(x-1)/(m-1): P:=[]:
for j from 1 to n do
  M:=Slope(j):
  h:=evalf(H*0.5/sqrt(1+M^2)):
  for k from 1 to m do
   y0:=Y(k): x0:=evalf(M-0.25+(y0-0.5)^2):
   if x0<a or x0>b then next: fi:
   P:=P,[[x0-h,y0-h*M],[x0+h,y0+h*M]]:
  od:
od:
Data:=[P[2..-1]]: opts:=color=BLACK,axes=FRAMED:
Plot1:=plot(Data,x=a..b,y=c..d,opts):
with(plots):
eq:={seq((y-0.5)^2=x-Slope(j)+1/4,j=1..n)}:
Plot2:=implicitplot(eq,x=a..b,y=c..d):
display([Plot1,Plot2]);
```

Exercises 1.7

Uniform Grid Method. Apply the uniform grid method as in Example 1, page 72 to make a direction field of 11×11 points for the given differential equation on $-1 \le x \le 1$, $-2 \le y \le 2$.

1. $y' = x + y(2-y)$
2. $y' = x + y(1-2y)$
3. $y' = 1 + y(2 \ldots$

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Due #11, #15

$\frac{dy}{dx} = x + y$ Sketch by the method of isoclines a direction field on $|x| \le 2$, $|y| \le 3$

$m = -3$ to 3, step $= 0.5$

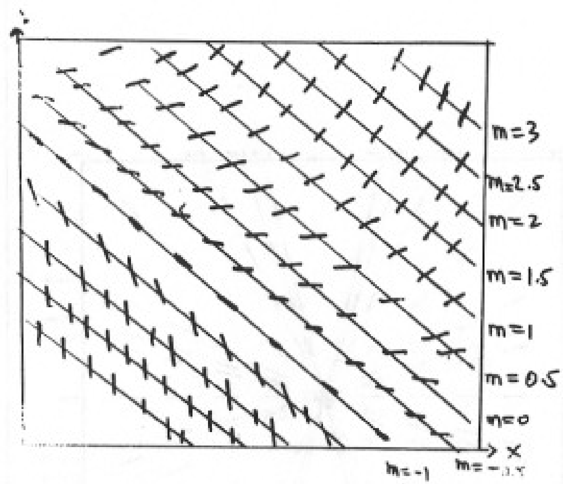
There are 13 slopes.

$x + y = m$ [isocline]

Def: an isocline is an equation $f(x,y) = m$ where $f(x,y) =$ RHS of the DE

$y - m = -x$

std form of a line, $y - y_0 = k(x - x_0)$



Each isocline is a straight line of slope −1.

segments lie atop the isocline!

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... to the slope.

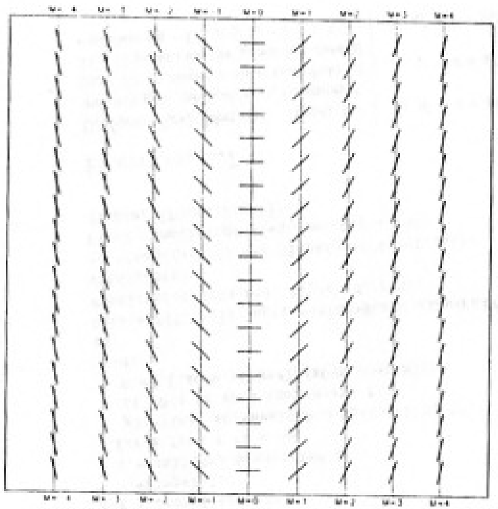
12) Given D.E.

Isocline is an equation f(x,y) = c where ... of the D.E.

Eq. of a line
$y = k(x - x_0)$

Each isocline is a straight line running vertically

Drawn via AutoCad



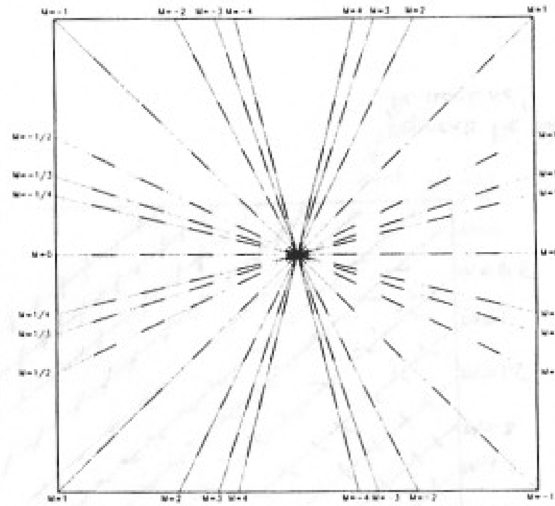
13)

$\frac{dy}{dx} = \frac{y}{x}$

$m = \frac{y}{x}$

$mx = y$

$m = -4$ to 4



Given D.E.

Same method as previous.