# 4.5 Undetermined Coefficients II

The study of undetermined coefficients continues. Recorded here are special methods for efficiently solving the easily-solved equations. **Linear algebra is not required** in any of the special methods: only calculus and college algebra are assumed as background.

The special methods provide a justification for the basic trial solution method of the preceding section.

## Library of Special Methods

It is assumed that the differential equation is already in easily-solved form: the library methods are designed to apply directly. If an equation requires a decomposition into easily-solved equations, then the desired solution is then the sum of the answers to the decomposed equations.

**Equilibrium and Quadrature Methods.** The special case of $ay'' + by' + cy = k$ where $k$ is a constant occurs so often that an efficient method has been isolated to find $y_p$. It is called the **equilibrium method**, because in the simplest case $y_p$ is a constant solution or an *equilibrium solution*. The method in words:

> Verify that the right side of the differential equation is constant. Cancel on the left side all derivative terms except for the lowest order and then solve for $y$ by quadrature.

The method works to find a solution, because if a derivative $y^{(n)}$ is constant, then all higher derivatives $y^{(n+1)}$, $y^{n+2}$, etc., are zero. A precise description follows for second order equations.

| Differential Equation | Cancelled DE | Particular Solution |
|---|---|---|
| $ay'' + by' + cy = k$, $c \neq 0$ | $cy = k$ | $y_p = \dfrac{k}{c}$ |
| $ay'' + by' = k$, $b \neq 0$ | $by' = k$ | $y_p = \dfrac{k}{b}x$ |
| $ay'' = k$, $a \neq 0$ | $ay'' = k$ | $y_p = \dfrac{k}{a}\dfrac{x^2}{2}$ |

The equilibrium method also applies to $n$th order linear differential equations $\sum_{i=0}^{n} a_i y^{(i)} = k$ with constant coefficients $a_0, \ldots, a_n$ and constant right side $k$.

A special case of the equilibrium method is the *simple quadrature method*, illustrated in Example **??**, page **??**. This method is used in elementary physics courses to solve falling body problems.

**The Polynomial Method.** The method applies to find a particular solution of $ay'' + by' + cy = p(x)$, where $p(x)$ represents a polynomial of degree $n \geq 1$. Such equations **always have a polynomial solution**; see Theorem **??**, page **??**.

Let $a$, $b$ and $c$ be given with $a \neq 0$. Differentiate the differential equation successively until the right side is constant:

$$(1) \qquad \begin{array}{llllllll} ay'' & + & by' & + & cy & = & p(x), \\ ay''' & + & by'' & + & cy' & = & p'(x), \\ ay^{iv} & + & by''' & + & cy'' & = & p''(x), \\ & & & & & \vdots & \\ ay^{(n+2)} & + & by^{(n+1)} & + & cy^{(n)} & = & p^{(n)}(x). \end{array}$$

Apply the equilibrium method to the *last equation* in order to find a polynomial **trial solution**

$$y(x) = d_m \frac{x^m}{m!} + \cdots + d_0.$$

It will emerge that $y(x)$ always has $n+1$ terms, but its degree can be either $n$, $n+1$ or $n+2$. The **undetermined coefficients** $d_0, \ldots, d_m$ are resolved by setting $x = 0$ in equations (**??**). The Taylor polynomial relations $d_0 = y(0), \ldots, d_m = y^{(m)}(0)$ give the equations

$$(2) \qquad \begin{array}{llllllll} ad_2 & + & bd_1 & + & cd_0 & = & p(0), \\ ad_3 & + & bd_2 & + & cd_1 & = & p'(0), \\ ad_4 & + & bd_3 & + & cd_2 & = & p''(0), \\ & & & & & \vdots & \\ ad_{n+2} & + & bd_{n+1} & + & cd_n & = & p^{(n)}(0). \end{array}$$

These equations can always be solved by **back-substitution**; linear algebra is not required. Three cases arise, according to the number of zero roots of the characteristic equation $ar^2 + br + c = 0$. The values $m = n, n+1, n+2$ correspond to zero, one or two roots $r = 0$.

**No root** $r = 0$. Then $c \neq 0$. There were $n$ integrations to find the trial solution, so $d_{n+2} = d_{n+1} = 0$. The unknowns are $d_0$ to $d_n$. The system can be solved by simple back-substitution to uniquely determine $d_0, \ldots, d_n$. The resulting polynomial $y(x)$ is the desired solution $y_p(x)$.

**One root** $r = 0$. Then $c = 0$, $b \neq 0$. The unknowns are $d_0, \ldots, d_{n+1}$. There is no condition on $d_0$; simplify the trial solution by taking $d_0 = 0$. Solve (**??**) for unknowns $d_1$ to $d_{n+1}$, as in the no root case.

**Double root** $r = 0$. Then $c = b = 0$ and $a \neq 0$. The equilibrium method gives a polynomial trial solution $y(x)$ involving $d_0, \ldots, d_{n+2}$. There are no conditions on $d_0$ and $d_1$. Simplify $y$ by taking $d_0 = d_1 = 0$. Solve (**??**) for unknowns $d_2$ to $d_{n+2}$, as in the no root case.

College algebra back-substitution applied to (**??**) is illustrated in Example **??**, page **??**. A complete justification of the polynomial method appears in the proof of Theorem **??**, page **??**.

### Recursive Polynomial Hybrid.

A *recursive method* based upon quadrature appears in Example **??**, page **??**. This method, independent from the *polynomial method* above, is useful when the number of equations in (**??**) is two or three.

Some researchers (see [Gupta]) advertise the recursive method as easy to remember, easy to use and faster than other methods. In this textbook, the method is advertised as a **hybrid**: equations in (**??**) are written down, but equations (**??**) are not. Instead, the undetermined coefficients are found recursively, by repeated quadrature and back-substitution.

Classroom testing of the recursive polynomial method reveals it is best suited to algebraic helmsmen with flawless talents. The method should be applied when conditions suggest rapid and reliable computation details. Error propagation possibilities dictate that polynomial solutions of degree 4 or larger be subjected to an answer check.

### Polynomial × Exponential Method.

The method applies to special equations $ay'' + by' + cy = p(x)e^{kx}$ where $p(x)$ is a polynomial. The idea, due to Kümmer, uses the transformation $y = e^{kx}Y$ to obtain the auxiliary equation

$$[a(D+k)^2 + b(D+k) + c]Y = p(x), \quad D = \frac{d}{dx}.$$

The polynomial method applies to find $Y$. Multiplication by $e^{kx}$ gives $y$.

Computational details are in Example **??**, page **??**. Justification appears in Theorem **??**. In words, to find the differential equation for $Y$:

> In the differential equation, replace $D$ by $D + k$ and cancel $e^{kx}$ on the RHS.

### Polynomial × Exponential × Cosine Method.

The method applies to equations $ay'' + by' + cy = p(x)e^{kx}\cos(mx)$ where $p(x)$ is a polynomial. Kümmer's transformation $y = e^{kx}\,\mathcal{R}\mathrm{e}(e^{imx}Y)$ gives the auxiliary problem

$$[a(D+z)^2 + b(D+z) + c]Y = p(x), \quad z = k + im, \quad D = \frac{d}{dx}.$$

The polynomial method applies to find $Y$. Symbol $\mathcal{R}\mathrm{e}$ extracts the real part of a complex number. Details are in Example **??**, page **??**. The formula is justified in Theorem **??**. In words, to find the equation for $Y$:

> In the differential equation, replace $D$ by $D + k + im$ and cancel $e^{kx}\cos mx$ on the RHS.

**Polynomial × Exponential × Sine Method.**
The method applies to equations $ay'' + by' + cy = p(x)e^{kx}\sin(mx)$ where $p(x)$ is a polynomial. Kümmer's transformation $y = e^{kx}\,\mathcal{I}\text{m}(e^{imx}Y)$ gives the auxiliary problem

$$[a(D+z)^2 + b(D+z) + c]Y = p(x), \quad z = k + im, \quad D = \frac{d}{dx}.$$

The polynomial method applies to find $Y$. Symbol $\mathcal{I}$m extracts the imaginary part of a complex number. Details are in Example **??**, page **??**. The formula is justified in Theorem **??**. In words, to find the equation for $Y$:

> In the differential equation, replace $D$ by $D + k + im$ and cancel $e^{kx}\sin mx$ on the RHS.

**Kümmer's Method.** The methods known above as the polynomial × exponential method, the polynomial × exponential × cosine method, and the polynomial × exponential × sine method, are collectively called **Kümmer's method**, because of their origin.

## Trial Solution Shortcut

The library of special methods leads to a justification for the **basic trial solution method**, a method which has been popularized by leading differential equation textbooks published over the past 50 years.

**How Kümmer's Method Predicts Trial Solutions.** Given $ay'' + by' + cy = f(x)$ where $f(x) =$(polynomial)$e^{kx}\cos mx$, then the method of Kümmer predicts $y = e^{kx}\,\mathcal{R}\text{e}\,(Y(x)(\cos mx + i\sin mx))$, where $Y(x)$ is a polynomial solution of a different, **associated differential equation**. In the simplest case, $Y(x) = \sum_{j=0}^{n} A_j x^j + i\sum_{j=0}^{n} B_j x^j$, a polynomial of degree $n$ with complex coefficients, matching the degree of the polynomial in $f(x)$. Expansion of the Kümmer formula for $y$ plus definitions $a_j = A_j - B_j$, $b_j = B_j + A_j$ gives a **trial solution**

$$y = \left(\cos(mx)\sum_{j=0}^{n} a_j x^j + \sin(mx)\sum_{j=0}^{n} b_j x^j\right) e^{kx}. \tag{3}$$

The undetermined coefficients are $a_0, \ldots, a_n, b_0, \ldots, b_n$. Exactly the same trial solution results when $f(x) =$(polynomial)$e^{kx}\sin mx$. If $m = 0$, then the trigonometric functions do not appear and the trial solution is either a polynomial ($k = 0$) or else a polynomial times an exponential.

The characteristic equation for the associated differential equation has root $r = 0$ exactly when $r = k + m\sqrt{-1}$ is a root of $ar^2 + br + c =$

0. Therefore, $Y$, and hence $y$, must be multiplied by $x$ for each time $k+m\sqrt{-1}$ is a root of $ar^2+br+c=0$. In the basic trial solution method, this requirement is met by multiplication by $x$ until the trial solution no longer contains a term of the homogeneous solution. Certainly both fixup rules produce exactly the same final trial solution.

Shortcuts using (**??**) have been known for some time. The results can be summarized in words as follows.

> If the right side of $ay''+by'+cy=f(x)$ is a polynomial of degree $n$ times $e^{kx}\cos(mx)$ or $e^{kx}\sin(mx)$, then an initial trial solution $y$ is given by relation (**??**), with undetermined coefficients $a_0, \ldots, a_n, b_0, \ldots, b_n$. Correct the trial solution $y$ by multiplication by $x$, once for each time $r=k+m\sqrt{-1}$ is a root of the characteristic equation $ar^2+br+c=0$.

**A Table Lookup Method.** The table below summarizes the form of an initial trial solution in special cases, according to the form of $f(x)$.

**Table 2. A Table Method for Trial Solutions.**
The table predicts the **initial trial solution** $y$ in the method of undetermined coefficients. Then the **fixup rule** below is applied to find the **final trial solution**. Symbol $n$ is the degree of the polynomial in column 1.

| Form of $f(x)$ | Values | Initial Trial Solution |
|---|---|---|
| constant | $k=m=0$ | $y=a_0$ =constant |
| polynomial | $k=m=0$ | $y=\sum_{j=0}^{n} a_j x^j$ |
| combination of $\cos mx$ and $\sin mx$ | $k=0$, $m>0$ | $y=a_0\cos mx+b_0\sin mx$ |
| (polynomial)$e^{kx}$ | $m=0$ | $y=\left(\sum_{j=0}^{n} a_j x^j\right)e^{kx}$ |
| (polynomial)$e^{kx}\cos mx$ or (polynomial)$e^{kx}\sin mx$ | $m>0$ | $y=\left(\sum_{j=0}^{n} a_j x^j\right)e^{kx}\cos mx + \left(\sum_{j=0}^{n} b_j x^j\right)e^{kx}\sin mx$ |

**The Fixup Rule.** The **final trial solution** is found by this rule:

> Given an initial trial solution $y$ for $au''+by'+cy=f(x)$, from Table **??**, correct $y$ by multiplication by $x$, once for each time that $r=k+m\sqrt{-1}$ is a root of the characteristic equation $ar^2+br+c=0$. This is equivalent to multiplication by $x$ until the trial solution no longer contains a term of the homogeneous solution.

Once the **final trial solution** $y$ is determined, then $y$ is substituted into the differential equation. The undetermined coefficients are found by matching terms of the form $x^j e^{kx}\cos(mx)$ and $x^j e^{kx}\sin(mx)$, which appear on the left and right side of the equation after substitution.

Details for lines 2-3 of Table **??** appear in Examples **??** and **??**.

**Alternate trial solution shortcut.** The method avoids the root testing of the fixup rule, at the expense of repeated substitutions. The simplicity of the method is appealing, but a few computations will convince the reader that the fixup rule is a more practical method.

> Let $y$ be the initial trial solution of Table ??. Substitute it into the differential equation. It will either compute $y_p$, or else some coefficients cannot be determined. In the latter case, multiply $y$ by $x$ and repeat, until a solution $y_p$ is found.

## Key theorems

The following results, whose proofs are delayed to page **??**, form the theoretical basis for the method of undetermined coefficients.

**Theorem 10 (Polynomial Solutions)**
Assume $a$, $b$, $c$ are constants, $a \neq 0$. Let $p(x)$ be a polynomial of degree $d$. Then $ay'' + by' + cy = p(x)$ has a polynomial solution $y$ of degree $d$, $d+1$ or $d+2$. Precisely, these three cases hold:

**Case 1**. $ay'' + by' + cy = p(x)$, $c \neq 0$. Then $y = y_0 + \cdots + y_d \dfrac{x^d}{d!}$.

**Case 2**. $ay'' + by' = p(x)$, $b \neq 0$. Then $y = \left(y_0 + \cdots + y_d \dfrac{x^d}{d!}\right) x$.

**Case 3**. $ay'' = p(x)$, $a \neq 0$. Then $y = \left(y_0 + \cdots + y_d \dfrac{x^d}{d!}\right) x^2$.

**Theorem 11 (Polynomial × Exponential)**
Assume $a$, $b$, $c$, $k$ are constants, $a \neq 0$, and $p(x)$ is a polynomial. If $Y$ is a solution of $[a(D+k)^2 + b(D+k) + c]Y = p(x)$, then $y = e^{kx}Y$ is a solution of $ay'' + by' + cy = p(x)e^{kx}$.

**Theorem 12 (Polynomial × Exponential × Cosine or Sine)**
Assume $a$, $b$, $c$, $k$, $m$ are real, $a \neq 0$, $m > 0$. Let $p(x)$ be a real polynomial and $z = k + im$. If $Y$ is a solution of $[a(D+z)^2 + b(D+z) + c]Y = p(x)$, then $y = e^{kx}\,\mathcal{R}\mathrm{e}(e^{imx}Y)$ is a solution of $ay'' + by' + cy = p(x)e^{kx}\cos(mx)$ and $y = e^{kx}\,\mathcal{I}\mathrm{m}(e^{imx}Y)$ is a solution of $ay'' + by' + cy = p(x)e^{kx}\sin(mx)$.

University courses usually assign the proofs of the key theorems as reading to save class time for examples.

**9 Example (Simple Quadrature)**
Solve for $y_p$ in $y'' = 2 - x + x^3$ using the fundamental theorem of calculus, verifying $y_p = x^2 - x^3/6 + x^5/20$.

**Solution**: Two integrations using the fundamental theorem of calculus give $y = y_0 + y_1 x + x^2 - x^3/6 + x^5/20$. The terms $y_0 + y_1 x$ represent the homogeneous solution $y_h$. Therefore, $y_p = x^2 - x^3/6 + x^5/20$ is reported. The method works in general for $ay'' + by' + cy = p(x)$, provided $b = c = 0$, that is, in **case 3** of Theorem **??**. Some explicit details:

| | |
|---|---|
| $\int y''(x)dx = \int (2 - x + x^3)dx$ | Integrate across on $x$. |
| $y' = y_1 + 2x - x^2/2 + x^4/4$ | Fundamental theorem. |
| $\int y'(x)dx = \int (y_1 + 2x - x^2/2 + x^4/4)dx$ | Integrate across again on $x$. |
| $y = y_0 + y_1 x + x^2 - x^3/6 + x^5/20$ | Fundamental theorem. |

**10 Example (Undetermined Coefficients: A Hybrid Method)**
Solve for $y_p$ in the equation $y'' - y' + y = 2 - x + x^3$ by the method of undetermined coefficients, verifying $y_p = -5 - x + 3x^2 + x^3$.

**Solution**: Let's begin by *calculating the trial solution* $y = d_0 + d_1 x + d_2 x^2/2 + x^3$. This is done by differentiation of $y'' - y' + y = 2 - x + x^3$ until the right side is constant:

$$y^v - y^{iv} + y''' = 6.$$

The equilibrium method solves the truncated equation $0 + 0 + y''' = 6$ by quadrature to give $y = d_0 + d_1 x + d_2 x^2/2 + x^3$.

The **undetermined coefficients** $d_0$, $d_1$, $d_2$ will be found by a classical technique in which the trial solution $y$ is back-substituted into the differential equation. We begin by computing the derivatives of $y$:

| | |
|---|---|
| $y = d_0 + d_1 x + d_2 x^2/2 + x^3$ | Calculated above; see Theorem **??**. |
| $y' = d_1 + d_2 x + 3x^2$ | Differentiate. |
| $y'' = d_2 + 6x$ | Differentiate again. |

The relations above are back-substituted into the differential equation $y'' - y' + y = 2 - x + x^3$ as follows:

$$\begin{aligned} 2 - x + x^3 &= y'' - y' + y && \text{Write the DE backwards.} \\ &= \quad [d_2 + 6x] \\ &\quad - [cd_1 + d_2 x + 3x^2] && \text{Substitute for } y,\ y',\ y''. \\ &\quad + [d_0 + d_1 x + d_2 x^2/2 + x^3] \\ &= \quad [c_2 - c_1 + c_0] \\ &\quad + [6 - d_2 + c_1]x \\ &\quad + [-3 + d_2/2]x^2 && \text{Collect on powers of } x. \\ &\quad + [1]x^3 \end{aligned}$$

The coefficients $d_0$, $d_1$, $d_2$ are found by **matching powers** on the LHS and RHS of the expanded equation:

$$(4)\qquad \begin{array}{rcll} 2 &=& [d_2 - d_1 + c_0] & \text{match constant term,} \\ -1 &=& [6 - d_2 + d_1] & \text{match } x\text{-term,} \\ 0 &=& [-3 + d_2/2] & \text{match } x^2\text{-term.} \end{array}$$

These equations are solved by back-substitution, starting with the last equation and proceeding to the first equation. The answers are successively $d_2 = 6$, $d_1 = -1$, $d_0 = -5$. For more detail on back-substitution, see the next example. Substitution into $y = d_0 + d_1x + d_2x^2/2 + x^3$ gives the particular solution $y_p = -5 - x + 3x^2 + x^3$.

## 11 Example (Undetermined Coefficients: Taylor's Method)

Solve for $y_p$ in the equation $y'' - y' + y = 2 - x + x^3$ by Taylor's method, verifying $y_p = -5 - x + 3x^2 + x^3$.

**Solution**: Theorem **??** implies that there is a polynomial solution $y = d_0 + d_1x + d_2x^2/2 + d_3x^3/6$. The **undetermined coefficients** $d_0$, $d_1$, $d_2$, $d_3$ will be found by a technique related to **Taylor's method** in calculus. The Taylor technique requires differential equations obtained by successive differentiation of $y'' - y' + y = 2 - x + x^3$, as follows.

| | |
|---|---|
| $y'' - y' + y = 2 - x + x^3$ | The original. |
| $y''' - y'' + y' = -1 + 3x^2$ | Differentiate the original once. |
| $y^{iv} - y''' + y'' = 6x$ | Differentiate the original twice. |
| $y^{v} - y^{iv} + y''' = 6$ | Differentiate the original three times. The process stops when the right side is constant. |

Set $x = 0$ in the above differential equations. Then substitute the Taylor polynomial derivative relations

$$y(0) = d_0,\ y'(0) = d_1,\ y''(0) = d_2,\ y'''(0) = d_3.$$

It is also true that $y^{iv}(0) = y^{v}(0) = 0$, since $y$ is a cubic. This produces the following equations for *undetermined coefficients* $d_0$, $d_1$, $d_2$, $d_3$:

$$\begin{array}{rcr} d_2 - d_1 + d_0 &=& 2 \\ d_3 - d_2 + d_1 &=& -1 \\ -d_3 + d_2 &=& 0 \\ d_3 &=& 6 \end{array}$$

These equations are solved by **back-substitution**, working in reverse order. No experience with linear algebra is required, because this is strictly a low-level college algebra method. Successive back-substitutions, working from the last equation in reverse order, give the answers

| | |
|---|---|
| $d_3 = 6,$ | Use the fourth equation first. |
| $d_2 = d_3$ | Solve for $d_2$ in the third equation. |
| $\quad = 6,$ | Back-substitute $d_3$. |
| $d_1 = -1 + d_2 - d_3$ | Solve for $d_1$ in the second equation. |

| | |
|---|---|
| $= -1,$ | Back-substitute $d_2$ and $d_3$. |
| $d_0 = 2 + d_1 - d_2$ | Solve for $d_0$ in the first equation. |
| $= -5.$ | Back-substitute $d_1$ and $d_2$. |

The result is $d_0 = -5$, $d_1 = -1$, $d_2 = 6$, $d_3 = 6$. Substitution into $y = d_0 + d_1 x + d_2 x^2/2 + d_3 x^3/6$ gives the particular solution $y_p = -5 - x + 3x^2 + x^3$.

## 12 Example (Polynomial Method: Recursive Hybrid)

In the equation $y'' - y' = 2 - x + x^3$, verify $y_p = -7x - 5x^2/2 - x^3 - x^4/4$ by the polynomial method, using a recursive hybrid.

**Solution**: A **recursive method** will be applied, based upon the fundamental theorem of calculus, as in Example **??**.

**Step 1**. Differentiate $y'' - y' = 2 - x + x^3$ until the right side is constant, to obtain

| | |
|---|---|
| Equation 1: $y'' - y' = 2 - x + x^3$ | The original. |
| Equation 2: $y''' - y'' = -1 + 3x^2$ | Differentiate the original once. |
| Equation 3: $y^{iv} - y''' = 6x$ | Differentiate the original twice. |
| Equation 4: $y^{v} - y^{iv} = 6$ | Differentiate the original three times. The process stops when the right side is constant. |

**Step 2**. There are 4 equations. Theorem **??** implies that there is a polynomial solution $y$ of degree 4. Then $y^v = 0$.

The last equation $y^v - y^{iv} = 6$ then gives $y^{iv} = -6$, which can be solved for $y'''$ by the fundamental theorem of calculus. Then $y''' = -6x + c$. Evaluate $c$ by requiring that $y$ satisfy equation 3: $y^{iv} - y''' = 6x$. Substitution of $y''' = -6x + c$, followed by setting $x = 0$ gives $-6 - c = 0$. Hence $c = -6$. The conclusion: $y''' = -6x - 6$.

**Step 3**. Solve $y''' = -6x - 6$, giving $y'' = -3x^2 - 6x + c$. Evaluate $c$ as in *Step 2* using equation 2: $y''' - y'' = -1 + 3x^2$. Then $-6 - c = -1$ gives $c = -5$. The conclusion: $y'' = -3x^2 - 6x - 5$.

**Step 4**. Solve $y'' = -3x^2 - 6x - 5$, giving $y' = -x^3 - 3x^2 - 5x + c$. Evaluate $c$ as in *Step 2* using equation 1: $y'' - y' = 2 - x + x^3$. Then $-5 - c = 2$ gives $c = -7$. The conclusion: $y' = -x^3 - 3x^2 - 5x - 7$.

**Step 5**. Solve $y' = -x^3 - 3x^2 - 5x - 7$, giving $y = -x^4/4 - x^3 - 5x^2/2 - 7x + c$. Just one solution is sought, so take $c = 0$. Then $y = -7x - 5x^2/2 - x^3 - x^4/4$. Theorem **??** also drops the constant term, because it is included in the homogeneous solution $y_h$. While this method duplicates all the steps in Example **??**, it remains attractive due to its simplistic implementation. The method is best appreciated when it terminates at step 2 or 3.

## 13 Example (Polynomial × Exponential)

Solve for $y_p$ in $y'' - y' + y = (2 - x + x^3)e^{2x}$, verifying that $y_p = e^{2x}(x^3/3 - x^2 + x + 1/3)$.

**Solution**: Let $y = e^{2x}Y$ and $[(D+2)^2 - (D+2) + 1]Y = 2 - x + x^3$, as per the *polynomial × exponential method*, page **??**. The equation $Y'' + 3Y' + 3Y = 2 - x + x^3$ will be solved by the polynomial method of Example **??**.

Differentiate $Y'' + 3Y' + 3Y = 2 - x + x^3$ until the right side is constant.

$$\begin{aligned}
&Y'' + 3Y' + 3Y = 2 - x + x^3\\
&Y''' + 3Y'' + 3Y' = -1 + 3x^2\\
&Y^{iv} + 3Y''' + 3Y'' = 6x\\
&Y^{v} + 3Y^{iv} + 3Y''' = 6
\end{aligned}$$

The last equation, by the equilibrium method, implies $Y$ is a polynomial of degree 4, $Y = d_0 + d_1x + d_2x^2/2 + d_3x^3/6$. Set $x = 0$ and $d_i = Y^{(i)}(0)$ in the preceding equations to get the system

$$\begin{aligned}
d_2 + 3d_1 + 3d_0 &= 2\\
d_3 + 3d_2 + 3d_1 &= -1\\
d_4 + 3d_3 + 3d_2 &= 0\\
d_5 + 3d_4 + 3d_3 &= 6
\end{aligned}$$

in which $d_4 = d_5 = 0$. Solving by back-substitution gives the answers $d_3 = 2$, $d_2 = -2$, $d_1 = 1$, $d_0 = 1/3$. Then $Y = x^3/3 - x^2 + x + 1/3$.

Finally, Kümmer's transformation $y = e^{2x}Y$ implies $y = e^{2x}(x^3/3 - x^2 + x + 1/3)$.

**14 Example (Polynomial × Exponential × Cosine)**
Solve in $y'' - y' + y = (3 - x)e^{2x}\cos(3x)$ for $y_p$, verifying that $y_p = \frac{1}{507}((26x - 107)e^{2x}\cos(3x) + (115 - 39x)e^{2x}\sin(3x))$.

**Solution**: Let $z = 2 + 3i$. If $Y$ satisfies $[(D+z)^2 - (D+z) + 1]Y = 3 - x$, then $y = e^{2x}\,\mathcal{R}\text{e}(e^{3ix}Y)$, by the method on page **??**. The differential equation simplifies into $Y'' + (3+6i)Y' + (9i-6)Y = 3 - x$. It will be solved by the recursion method of Example **??**.

**Step 1**. Differentiate $Y'' + (3+6i)Y' + (9i-6)Y = 3 - x$ until the right side is constant, to obtain $Y''' + (3+6i)Y'' + (9i-6)Y' = -1$. The conclusion: $Y' = 1/(6 - 9i)$.

**Step 2**. Solve $Y' = 1/(6-9i)$ for $Y = x/(6-9i) + c$. Evaluate $c$ by requiring $Y$ to satisfy the original equation $Y'' + (3+6i)Y' + (9i-6)Y = 3 - x$. Substitution of $Y' = x/(6-9i) + c$, followed by setting $x = 0$ gives $0 + (3+6i)/(6-9i) + (9i-6)c = 3$. Hence $c = (-15 + 33i)/(6 - 9i)^2$. The conclusion: $Y = x/(6 - 9i) + (-15 + 33i)/(6 - 9i)^2$.

**Step 3**. Use variable $y = e^{2x}\,\mathcal{R}\text{e}(e^{3ix}Y)$ to complete the solution. This is the point where complex arithmetic must be used. Let $y = e^{2x}\mathcal{Y}$ where $\mathcal{Y} = \mathcal{R}\text{e}(e^{3ix}Y)$. Some details:

$$\begin{aligned}
Y &= \frac{x}{6-9i} + \frac{-15+33i}{(6-9i)^2} && \text{The plan: write } Y = Y_1 + iY_2.\\
&= x\frac{6+9i}{6^2+9^2} + \frac{(-15+33i)(6+9i)^2}{(6^2+9^2)^2} && \text{Use } 1/Z = \overline{Z}/|Z|^2,\ Z = a+ib,\ \overline{Z} = a - ib,\ |Z| = a^2 + b^2.\\
&= \frac{2x}{39} + \frac{xi}{13} + \frac{-2889 - 3105i}{117^2} && \text{Use } 6^2 + 9^2 = 117 = (9)(13).
\end{aligned}$$

$$= \frac{26x-107}{507} + i\frac{39x-115}{507} \qquad \text{Split off real and imaginary.}$$

$$Y_1 = \frac{26x-107}{507}, \quad Y_2 = \frac{39x-115}{507} \qquad \text{Decomposition found.}$$

$$\begin{aligned}
\mathcal{Y} &= \mathcal{R}\text{e}((\cos 3x + i\sin 3x)(Y_1+iY_2)) && \text{Use } e^{3ix} = \cos 3x + i\sin 3x.\\
&= Y_1\cos 3x - Y_2 \sin 3x && \text{Take the real part.}\\
&= \frac{26x-107}{507}\cos 3x + \frac{115-39x}{507}\sin 3x && \text{Substitute for } Y_1,\ Y_2.
\end{aligned}$$

The solution $y = e^{2x}\mathcal{Y}$ multiplies the above display by $e^{2x}$. This verifies the formula $y_p = \frac{1}{507}((26x-107)e^{2x}\cos(3x) + (115-39x)e^{2x}\sin(3x))$.

**15 Example (Polynomial × Exponential × Sine)**
Solve in $y'' - y' + y = (3-x)e^{2x}\sin(3x)$ for $y_p$, verifying that a particular solution is $y_p = \frac{1}{507}\left((39x-115)e^{2x}\cos(3x) + (26x-107)e^{2x}\sin(3x)\right)$.

**Solution**: Let $z = 2+3i$. Kümmer's transformation $y = e^{2x}\,\mathcal{I}\text{m}(e^{3ix}Y)$ as on page **??** implies that $Y$ satisfies $[(D+z)^2 - (D+z)+1]Y = 3-x$. This equation has been solved in the previous example: $Y = Y_1 + iY_2$ with $Y_1 = (26x-107)/507$ and $Y_2 = (39x-115)/507$. Let $\mathcal{Y} = \mathcal{I}\text{m}(e^{3ix}Y)$. Then

$$\begin{aligned}
\mathcal{Y} &= \mathcal{I}\text{m}((\cos 3x + i\sin 3x)(Y_1+iY_2)) && \text{Expand complex factors.}\\
&= Y_2\cos 3x + Y_1\sin 3x && \text{Extract the imaginary part.}\\
&= \frac{(39x-115)\cos 3x + (26x-107)\sin 3x}{507} && \text{Substitute for } Y_1 \text{ and } Y_2.
\end{aligned}$$

The solution $y = e^{2x}\mathcal{Y}$ multiplies the display by $e^{2x}$. This verifies the formula $y = \frac{1}{507}\left((39x-115)e^{2x}\cos(3x) + (26x-107)e^{2x}\sin(3x)\right)$.

**16 Example (Undetermined Coefficient Algorithm: Library Methods)**
Solve $y'' - y' + y = 1 + e^x + \cos(x)$, verifying $y = c_1e^{x/2}\cos(\sqrt{3}x/2) + c_2e^{x/2}\sin(\sqrt{3}x/2) + 1 + e^x - \sin(x)$.

**Solution**: There are $n = 3$ easily solved equations: $y_1'' - y_1' + y_1 = 1$, $y_2'' - y_2' + y_2 = e^x$ and $y_3'' - y_3' + y_3 = \cos(x)$. The plan is that each such equation is solvable by one of the **library methods**. Then $y_p = y_1 + y_2 + y_3$ is the sought particular solution.

**Equation 1:** $y_1'' - y_1' + y_1 = 1$. It is solved by *the equilibrium method*, which gives immediately solution $y_1 = 1$.

**Equation 2:** $y_2'' - y_2' + y_2 = e^x$. Then $y_2 = e^xY$ and $[(D+1)^2 - (D+1) + 1]Y = 1$, by the *polynomial × exponential method*. The equation simplifies to $Y'' + Y' + Y = 1$. Obtain $Y = 1$ by the *equilibrium method*, then $y_2 = e^x$.

**Equation 3:** $y_3'' - y_3' + y_3 = \cos(x)$. Then $[(D+i)^2 - (D+i) + 1]Y = 1$ and $y_3 = \mathcal{R}\text{e}(e^{ix}Y)$, by the *polynomial × exponential × cosine method*. The equation simplifies to $Y'' + (2i-1)Y' - iY = 1$. Obtain $Y = i$ by the *equilibrium method*. Then $y_3 = \mathcal{R}\text{e}(e^{ix}Y)$ implies $y_3 = -\sin(x)$.

**Solution** $y_p$. The particular solution is given by addition, $y_p = y_1 + y_2 + y_3$. Therefore, $y_p = 1 + e^x - \sin(x)$.

**Solution** $y_h$. The homogeneous solution $y_h$ is the linear equation *recipe* solution for $y'' - y' + y = 0$, which uses the characteristic equation $r^2 - r + 1 = 0$. The latter has roots $r = (1 \pm i\sqrt{3})/2$ and then $y_h = c_1 e^{x/2}\cos(\sqrt{3}x/2) + c_2 e^{x/2}\sin(\sqrt{3}x/2)$ where $c_1$ and $c_2$ are arbitrary constants.

**General Solution**. Add $y_h$ and $y_p$ to obtain the general solution

$$y = c_1 e^{x/2}\cos(\sqrt{3}x/2) + c_2 e^{x/2}\sin(\sqrt{3}x/2) + 1 + e^x - \sin(x).$$

**17 Example (Sine–Cosine Trial solution)** Verify for $y'' + 4y = \sin x - \cos x$ that $y_p(x) = 5\cos x + 3\sin x$.

**Solution**: The lookup table method suggests to substitute $y = d_1 \cos x + d_2 \sin x$ into the differential equation. The fixup rule does not apply, because the homogeneous solution terms involve $\cos 2x$, $\sin 2x$. Use $u'' = -u$ for $u = \sin x$ or $u = \cos x$ to obtain the relation

$$\begin{aligned}\sin x - \cos x &= y'' + 4y \\ &= (-d_1 + 4)\cos x + (-d_2 + 4)\sin x.\end{aligned}$$

Comparing sides, matching sine and cosine terms, gives

$$\begin{aligned}-d_1 + 4 &= -1, \\ -d_2 + 4 &= 1.\end{aligned}$$

Solving, $d_1 = 5$ and $d_2 = 3$. The trial solution $y = d_1 \cos x + d_2 \sin x$ becomes $y_p(x) = 5\cos x + 3\sin x$.

**Historical Notes.** The method of undetermined coefficients was presented using the idea of a **trial solution**; see the **Basic Trial Solution Method**, Page 174. Textbooks that present this method appear in the references, especially [**?**] and [**?**].

If $f(x)$ is a polynomial, then the trial solution is a polynomial $y = d_0 + \cdots + d_k x^k$ with unknown coefficients. It is substituted into the non-homogeneous differential equation to determine the coefficients $d_0, \ldots,$ $d_k$, as in Example **??**. The Taylor method in Example **??** implements the same ideas. In the some textbook presentations, the three key theorems of this section are replaced by Table **??** and the **Fixup Rule** on page **??**. Attempts have been made to integrate the fixup rule into the table itself; see [**?**] and [**?**].

The *method of annihilators* has been used as an alternative approach; see [**?**]. The approach gives a deeper insight into higher order differential equations. It requires substantial knowledge of linear algebra and differential operator calculus.

The idea to employ a recursive polynomial method seems to appear first in a paper by Love [**?**]. A generalization and expansion of details appears in [Gupta]. The method is certainly worth learning, but doing so does not excuse the reader from also learning other methods. The recursive method is a worthwhile hybrid method for special circumstances.

**Proof of Theorem ??:** The three cases correspond to zero, one or two roots $r = 0$ for the characteristic equation $ar^2 + br + c = 0$. The missing constant and $x$-terms in **case 2** and **case 3** are justified by including them in the homogeneous solution $y_h$, instead of in the particular solution $y_p$.

Assume $p(x)$ has degree $d$ and succinctly write down the successive derivatives of the differential equation as

$$ay^{(2+k)} + by^{(1+k)} + cy^{(k)} = p^{(k)}(x), \quad k = 0, \ldots, d. \tag{5}$$

Assume, to consider simultaneously all three cases, that

$$y = y_0 + y_1 + \cdots + y_{m+d}\frac{x^{m+d}}{(m+d)!}$$

where $m = 0, 1, 2$ corresponding to cases 1,2,3, respectively. It has to be shown that there are coefficients $y_0, \ldots, y_{m+d}$ such that $y$ is a solution of $ay'' + by' + cy = p(x)$.

Let $x = 0$ in equations (**??**) and use the definition of polynomial $y$ to obtain the equations

$$ay_{2+k} + by_{1+k} + cy_k = p^{(k)}(0), \quad k = 0, \ldots, d. \tag{6}$$

In **case 1** ($c \neq 0$), $m = 0$ and the last equation in (**??**) gives $y_{m+d} = p^{(d)}(0)/c$. Back-substitution succeeds in finding the other coefficients, in reverse order, because $y^{(d+1)}(0) = y^{(d+2)}(0) = 0$, in this case. *Define* the constants $y_0$ to $y_d$ to be the solutions of (**??**). Define $y_{d+1} = y_{d+2} = 0$.

In **case 2** ($c = 0$, $b \neq 0$), $m = 1$ and the last equation in (**??**) gives $y_{m+d} = p^{(d)}(0)/b$. Back-substitution succeeds in finding the other coefficients, in reverse order, because $y^{(d+2)}(0) = 0$, in this case. However, $y_0$ is undetermined. Take it to be zero, then *define* $y_1$ to $y_{d+1}$ to be the solutions of (**??**). Define $y_{d+2} = 0$.

In **case 3** ($c = b = 0$), $m = 2$ and the last equation in (**??**) gives $y_{m+d} = p^{(d)}(0)/a$. Back-substitution succeeds in finding the other coefficients, in reverse order. However, $y_0$ and $y_1$ are undetermined. Take them to be zero, then *define* $y_2$ to $y_{d+2}$ to be the solutions of (**??**).

It remains to prove that the polynomial $y$ so defined is a solution of the differential equation $ay'' + by' + cy = p(x)$. Begin by applying quadrature to the last differentiated equation $ay^{(2+d)} + by^{(1+d)} + cy^{(d)} = p^{(d)}(x)$. The result is $ay^{(1+d)} + by^{(d)} + cy^{(d-1)} = p^{(d-1)}(x) + C$ with $C$ undetermined. Set $x = 0$ in this equation. Then relations (**??**) say that $C = 0$. This process can be continued until $ay'' + by' + cy = q(x)$ is obtained, hence $y$ is a solution.

**Proof of Theorem ??:** Kümmer's transformation $y = e^{kx}Y$ is differentiated twice to give the formulas

$$\begin{aligned} y &= e^{kx}Y, \\ y' &= ke^{kx}Y + e^{kx}Y' \\ &= e^{kx}(D+k)Y, \\ y'' &= k^2e^{kx}Y + 2ke^{kx}Y' + e^{kx}Y'' \\ &= e^{kx}(D+k)^2Y. \end{aligned}$$

Insert them into the differential equation $a(D+k)^2Y+b(D+k)Y+cY=p(x)$. Then multiply through by $e^{kx}$ to remove the common factor $e^{-kx}$ on the left, giving $ay''+by'+cy=p(x)e^{kx}$. This completes the proof.

**Proof of Theorem ??:** Abbreviate $ay''+by'+cy$ by $Ly$. Consider the complex equation $Lu=p(x)e^{zx}$, to be solved for $u=u_1+iu_2$. According to Theorem **??**, $u$ can be computed as $u=e^{zx}Y$ where $[a(D+z)^2+b(D+z)+c]Y=p(x)$. Take the real and imaginary parts of $u=e^{zx}Y$ and $Lu=p(x)e^{zx}$. Then $u_1=\mathcal{R}\mathrm{e}(e^{zx}Y)$ and $u_2=\mathcal{I}\mathrm{m}(e^{zx}Y)$ satisfy $Lu_1=\mathcal{R}\mathrm{e}(p(x)e^{zx})=p(x)\cos(mx)e^{kx}$ and $Lu_2=\mathcal{I}\mathrm{m}(p(x)e^{zx})=p(x)\sin(mx)e^{kx}$. This completes the proof.

## Exercises 4.5

Polynomial Solutions. Determine a polynomial solution $y_p$ for the given differential equation. Apply Theorem **??**, page **??**, and model the solution after Examples **??**, **??**, **??** and **??**.

**1.** $y''=x$

**2.** $y''=x-1$

**3.** $y''=x^2-x$

**4.** $y''=x^2+x-1$

**5.** $y''-y'=1$

**6.** $y''-5y'=10$

**7.** $y''-y'=x$

**8.** $y''-y'=x-1$

**9.** $y''-y'+y=1$

**10.** $y''-y'+y=-2$

**11.** $y''+y=1-x$

**12.** $y''+y=2+x$

**13.** $y''-y=x^2$

**14.** $y''-y=x^3$

Polynomial-Exponential Solutions. Determine a solution $y_p$ for the given differential equation. Apply Theorem **??**, page **??**, and model the solution after Example **??**.

**15.** $y''+y=e^x$

**16.** $y''+y=e^{-x}$

**17.** $y''=e^{2x}$

**18.** $y''=e^{-2x}$

**19.** $y''-y=(x+1)e^{2x}$

**20.** $y''-y=(x-1)e^{-2x}$

**21.** $y''-y'=(x+3)e^{2x}$

**22.** $y''-y'=(x-2)e^{-2x}$

**23.** $y''-3y'+2y=(x^2+3)e^{3x}$

**24.** $y''-3y'+2y=(x^2-2)e^{-3x}$

Sine and Cosine Solutions. Determine a solution $y_p$ for the given differential equation. Apply Theorem **??**, page **??**, and model the solution after Examples **??** and **??**.

**25.** $y''=\sin(x)$

**26.** $y''=\cos(x)$

**27.** $y''+y=\sin(x)$

**28.** $y''+y=\cos(x)$

**29.** $y''=(x+1)\sin(x)$

**30.** $y''=(x+1)\cos(x)$

**31.** $y''-y=(x+1)e^x\sin(2x)$

**32.** $y''-y=(x+1)e^x\cos(2x)$

**33.** $y''-y'-y=(x^2+x)e^x\sin(2x)$

**34.** $y''-y'-y=(x^2+x)e^x\cos(2x)$

Undetermined Coefficients Algorithm. Determine a solution $y_p$ for the given differential equation. Apply the polynomial algorithm, page **??**, and model the solution after Example **??**.

**35.** $y'' = x + \sin(x)$

**36.** $y'' = 1 + x + \cos(x)$

**37.** $y'' + y = x + \sin(x)$

**38.** $y'' + y = 1 + x + \cos(x)$

**39.** $y'' + y = \sin(x) + \cos(x)$

**40.** $y'' + y = \sin(x) - \cos(x)$

**41.** $y'' = x + xe^x + \sin(x)$

**42.** $y'' = x - xe^x + \cos(x)$

**43.** $y'' - y = \sinh(x) + \cos^2(x)$

**44.** $y'' - y = \cosh(x) + \sin^2(x)$

**45.** $y'' + y' - y = x^2e^x + xe^x\cos(2x)$

**46.** $y'' + y' - y = x^2e^{-x} + xe^x\sin(2x)$

Additional Proofs. The exercises below fill in details in the text.

**49.** **(Theorem ??)** Supply the details in the proof of Theorem **??** for case 1. In particular, give the details for back-substitution.

**50.** **(Theorem ??)** Supply the details in the proof of Theorem **??** for case 2. In particular, give the details for back-substitution and explain fully why it is possible to select $y_0 = 0$.

**51.** **(Theorem ??)** Supply the details in the proof of Theorem **??** for case 3. In particular, explain why back-substitution leaves $y_0$ and $y_1$ undetermined, and why it is possible to select $y_0 = y_1 = 0$.