Finding a Separable Form

Given differential equation $y' = f(x,y)$, invent values x_0, y_0 such that $f(x_0,y_0) \neq 0$. Define F, G by the formulas

$$F(x) = \frac{f(x,y_0)}{f(x_0,y_0)}, \quad G(y) = f(x_0,y). \tag{1}$$

Because $f(x_0,y_0) \neq 0$, then (1) makes sense. Test I *infra* implies the following practical test.

Theorem 3 (Separability Test)
Let F and G be defined by (1). Multiply FG. Then

(a) If $F(x)G(y) = f(x,y)$, then $y' = f(x,y)$ is **separable**.

(b) If $F(x)G(y) \neq f(x,y)$, then $y' = f(x,y)$ is **not separable**.

Invention and Application. Initially, let (x_0,y_0) be $(0,0)$ or $(1,1)$ or some suitable pair, for which $f(x_0,y_0) \neq 0$; then define F and G by (1). Multiply to test the equation $FG = f$.

The algebra will discover a factorization $f = F(x)G(y)$ without having to know algebraic tricks like factorizing multi-variable equations. But if $FG \neq f$, then the algebra *proves* the equation is not separable.

Non-Separability Tests. The first test uses the relation

$$f(x, y_0)f(x_0, y) - f(x_0, y_0)f(x, y) \neq 0. \quad (2)$$

Test I	Equation $y' = f(x, y)$ is not separable if for some pair of points (x_0, y_0), (x, y) in the domain of f, (2) holds.
Test II	The equation $y' = f(x, y)$ is not separable if either $f_x(x, y)/f(x, y)$ is non-constant in y or $f_y(x, y)/f(x, y)$ is non-constant in x.

Test I details. Assume $f(x, y) = F(x)G(y)$, then equation (2) fails because each term on the left side of (2) equals $F(x)G(y_0)F(x_0)G(y)$ for all choices of (x_0, y_0) and (x, y) (hence contradiction $0 \neq 0$).

Test II details. Assume $f(x, y) = F(x)G(y)$ and F, G are sufficiently differentiable. Then $f_x(x, y)/f(x, y) = F'(x)/F(x)$ is independent of y and $f_y(x, y)/f(x, y) = G'(y)/G(y)$ is independent of x.

Illustration.

Consider $y' = xy + y^2$.

Test I implies it is not separable, because the left side of the relation is

$$\begin{aligned}\text{LHS} &= f(x,1)f(0,y) - f(0,1)f(x,y) \\ &= (x+1)y^2 - (xy + y^2) \\ &= x(y^2 - y) \\ &\neq 0.\end{aligned}$$

Test II implies it is not separable, because

$$\frac{f_x}{f} = \frac{1}{x+y}$$

is not constant as a function of y.

Variables-Separable Method

The method determines two kinds of solution formulas.

Equilibrium Solutions. They are the constant solutions $y = c$ of $y' = f(x, y)$. For any equation, find them by substituting $y = c$ into the differential equation.

Non-Equilibrium Solutions. For a separable equation

$$y' = F(x)G(y),$$

a non-equilibrium solution y is a solution with $G(y) \neq 0$. It is found by dividing by $G(y)$, then applying the method of quadrature.

Finding Non-Equilibrium Solutions

A given solution $y(x)$ satisfying $G(y(x)) \neq 0$ throughout its domain of definition is called a non-equilibrium solution. Then division by $G(y(x))$ is allowed.

The *method of quadrature* applies to the separated equation $y'/G(y(x)) = F(x)$. Some details:

$\int_{x_0}^{x} \frac{y'(t)dt}{G(y(t))} = \int_{x_0}^{x} F(t)dt$	Integrate both sides of the separated equation over $x_0 \leq t \leq x$.
$\int_{y_0}^{y(x)} \frac{du}{G(u)} = \int_{x_0}^{x} F(t)dt$	Apply on the left the change of variables $u = y(t)$. Define $y_0 = y(x_0)$.
$y(x) = W^{-1}\left(\int_{x_0}^{x} F(t)dt\right)$	Define $W(y) = \int_{y_0}^{y} du/G(u)$. Take inverses to isolate $y(x)$.

In practise, the last step with W^{-1} is never done. The preceding formula is called the *implicit solution*. Some work is done to find algebraically an *explicit solution*, as is given by W^{-1}.

Explicit and Implicit Solutions

Definition 2 (Explicit Solution)
A solution of $y' = f(x, y)$ is called **explicit** provided it is given by an equation

$$y = \text{an expression independent of } y.$$

To elaborate, on the left side must appear exactly the symbol y followed by an equal sign. Symbols y and $=$ are followed by an expression which does not contain the symbol y.

Definition 3 (Implicit Solution)
A solution of $y' = f(x, y)$ is called **implicit** provided it is not explicit.

The variables-separable method gives equilibrium solutions $y = c$, which are already *explicit*. Equations like $2y = x$ are not explicit (they are called *implicit*) because the coefficient of y on the left is not 1. Similarly, $y = x + y^2$ is not explicit because the right side contains symbol y. Equation $y = e^{\pi}$ is explicit because the right side fails to contain symbol y (symbol x may be absent). Applications can leave the non-equilibrium solutions in *implicit* form $\int_{y_0}^{y(x)} du/G(u) = \int_{x_0}^{x} F(t)dt$, with serious effort being expended to do the indicated integrations.