5.8 Resonance

The study of vibrating mechanical systems ends here with the theory of pure and practical resonance.

Pure Resonance

The notion of **pure resonance** in the differential equation

$$x''(t) + \omega_0^2\, x(t) = F_0 \cos(\omega t) \tag{1}$$

is the existence of a solution that is unbounded as $t \to \infty$. We already know (page 224) that for $\omega \neq \omega_0$, the general solution of (1) is the sum of two harmonic oscillations, hence it is bounded. Equation (1) for $\omega = \omega_0$ has by the method of undetermined coefficients the unbounded oscillatory solution $x(t) = \dfrac{F_0}{2\omega_0}\, t \,\sin(\omega_0\, t)$. To summarize:

> Pure resonance occurs exactly when the natural internal frequency ω_0 matches the natural external frequency ω, in which case all solutions of the differential equation are unbounded.

In Figure 20, this is illustrated for $x''(t) + 16x(t) = 8 \cos 4t$, which in (1) corresponds to $\omega = \omega_0 = 4$ and $F_0 = 8$.

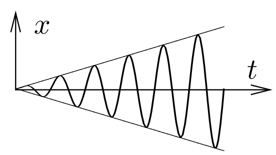

Figure 20. Pure resonance for $x''(t) + 16x(t) = 8 \cos \omega t$. Graphed are the solution $x(t) = t \sin 4t$ for $\omega = 4$ and the envelope curves $x = \pm t$.

Resonance and undetermined coefficients. An explanation of resonance can be based upon the theory of undetermined coefficients. An initial trial solution of

$$x''(t) + 16x(t) = 8 \cos \omega t$$

is $x = d_1 \cos \omega t + d_2 \sin \omega t$. The homogeneous solution $x_h = c_1 \cos 4t + c_2 \sin 4t$ considered in the **fixup rule** has duplicate terms exactly when the natural frequencies match: $\omega = 4$. Then the final trial solution is

$$x(t) = \begin{cases} d_1 \cos \omega t + d_2 \sin \omega t & \omega \neq 4, \\ t(d_1 \cos \omega t + d_2 \sin \omega t) & \omega = 4. \end{cases} \tag{2}$$

Even before the undetermined coefficients d_1, d_2 are evaluated, we can decide that unbounded solutions occur exactly when frequency matching

$\omega = 4$ occurs, because of the amplitude factor t. If $\omega \neq 4$, then $x_p(t)$ is a pure harmonic oscillation, hence bounded. If $\omega = 4$, then $x_p(t)$ equals a time–varying amplitude Ct times a pure harmonic oscillation, hence it is unbounded.

The Wine Glass Experiment. Equation (1) is advertised as the basis for a physics experiment which has appeared often on Public Television, called the *wine glass experiment*. A famous physicist, in front of an audience of physics students, equips a lab table with a frequency generator, an amplifier and an audio speaker. The *valuable* wine glass is replaced by a glass beaker. The frequency generator is tuned to the natural frequency of the glass beaker ($\omega \approx \omega_0$), then the volume knob on the amplifier is suddenly turned up (F_0 adjusted larger), whereupon the sound waves emitted from the speaker break the glass beaker.

The glass itself will vibrate at a certain frequency, as can be determined experimentally by *pinging* the glass rim. This vibration operates within elastic limits of the glass and the glass will not break under these circumstances. A physical explanation for the breakage is that an incoming sound wave from the speaker is timed to add to the glass rim excursion. After enough amplitude additions, the glass rim moves beyond the elastic limit and the glass breaks. The explanation implies that the external frequency from the speaker has to match the natural frequency of the glass. But there is more to it: the glass has some natural damping that nullifies feeble attempts to increase the glass rim amplitude. The physicist uses to great advantage this natural damping to *tune* the external frequency to the glass. The reason for turning up the volume on the amplifier is to nullify the damping effects of the glass. The amplitude additions then build rapidly and the glass breaks.

Soldiers Breaking Cadence. The collapse of the Broughton bridge near Manchester, England in 1831 is blamed for the now–standard practise of breaking cadence when soldiers cross a bridge. Bridges like the Broughton bridge have many natural low frequencies of vibration, so it is possible for a column of soldiers to vibrate the bridge at one of the bridge's natural frequencies. The bridge locks onto the frequency while the soldiers continue to add to the excursions with every step, causing larger and larger bridge oscillations.

Practical Resonance

The notion of pure resonance is easy to understand both mathematically and physically, because frequency matching characterizes the event. This ideal situation never happens in the physical world, because *damping is* *always present.* In the presence of damping $c > 0$, it will be established below that *only bounded solutions exist* for the forced spring-mass system

$$mx''(t) + cx'(t) + kx(t) = F_0 \cos \omega t. \tag{3}$$

Our intuition about resonance seems to vaporize in the presence of damping effects. But not completely. Most would agree that the undamped intuition is correct when the damping effects are nearly zero.

Practical resonance is said to occur when the external frequency ω has been tuned to produce the largest possible solution (a more precise definition appears below). It will be shown that this happens for the condition

$$\omega = \sqrt{k/m - c^2/(2m^2)}, \quad k/m - c^2/(2m^2) > 0. \tag{4}$$

Pure resonance $\omega = \omega_0 \equiv \sqrt{k/m}$ is the limiting case obtained by setting the damping constant c to zero in condition (4). This strange but predictable interaction exists between the damping constant c and the size of solutions, relative to the external frequency ω, even though all solutions remain bounded.

The decomposition of $x(t)$ into homogeneous solution $x_h(t)$ and particular solution $x_p(t)$ gives some intuition into the complex relationship between the input frequency ω and the size of the solution $x(t)$.

The homogeneous solution. For positive damping, $c > 0$, equation (3) has homogeneous solution $x_h(t) = c_1 x_1(t) + c_2 x_2(t)$ where according to the *recipe* the basis elements x_1 and x_2 are given in terms of the roots of the characteristic equation $mr^2 + cr + k = 0$, as classified by the discriminant $D = c^2 - 4mk$, as follows:

Case 1, $D > 0$	$x_1 = e^{r_1 t}$, $x_2 = e^{r_2 t}$ with r_1 and r_2 negative.
Case 2, $D = 0$	$x_1 = e^{r_1 t}$, $x_2 = te^{r_1 t}$ with r_1 negative.
Case 3, $D < 0$	$x_1 = e^{\alpha t} \cos \beta t$, $x_2 = e^{\alpha t} \sin \beta t$ with $\beta > 0$ and α negative.

It follows that $x_h(t)$ contains a negative exponential factor, regardless of the positive values of m, c, k. A solution $x(t)$ is called a **transient solution** provided it satisfies the relation $\lim_{t\to\infty} x(t) = 0$. The conclusion:

> The homogeneous solution $x_h(t)$ of the equation $mx''(t) + cx'(t) + kx(t) = 0$ is a transient solution for all positive values of m, c, k.

A transient solution graph $x(t)$ for large t lies atop the axis $x = 0$, as in Figure 21, because $\lim_{t\to\infty} x(t) = 0$.

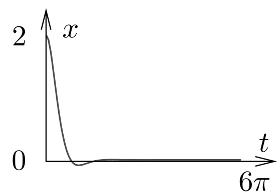

Figure 21. Transient oscillatory solution $x = 2e^{-t}(\cos t + \sin t)$ of the differential equation $x'' + 2x' + 2x = 0$.

The particular solution. The method of undetermined coefficients gives a trial solution of the form $x(t) = A\cos\omega t + B\sin\omega t$ with coefficients A, B satisfying the equations

$$
\begin{array}{rcrcl}
(k - m\omega^2)A & + & (c\omega)B & = & F_0, \\
(-c\omega)A & + & (k - m\omega^2)B & = & 0.
\end{array} \tag{5}
$$

Solving (5) with Cramer's rule or elimination produces the solution

$$
A = \frac{(k - m\omega^2)F_0}{(k - m\omega^2)^2 + (c\omega)^2}, \quad B = \frac{c\omega F_0}{(k - m\omega^2)^2 + (c\omega)^2}. \tag{6}
$$

The **steady–state** solution, periodic of period $2\pi/\omega$, is given by

$$
\begin{aligned}
x_p(t) &= \frac{F_0}{(k - m\omega^2)^2 + (c\omega)^2}\Big((k - m\omega^2)\cos\omega t + (c\omega)\sin\omega t\Big) \\
&= \frac{F_0}{\sqrt{(k - m\omega^2)^2 + (c\omega)^2}}\cos(\omega t - \alpha),
\end{aligned} \tag{7}
$$

where α is defined by the phase–amplitude relations (see page 216)

$$
\begin{aligned}
&C\cos\alpha = k - m\omega^2, \quad C\sin\alpha = c\omega, \\
&C = F_0/\sqrt{(k - m\omega^2)^2 + (c\omega)^2}.
\end{aligned} \tag{8}
$$

The terminology **steady–state** refers to that part $x_{\rm SS}(t)$ of the solution $x(t)$ that remains when the transient portion is removed, that is, when all terms containing negative exponentials are removed. As a result, for large T, the graphs of $x(t)$ and $x_{\rm SS}(t)$ on $t \ge T$ are the same. This feature of $x_{\rm SS}(t)$ allows us to find its graph directly from the graph of $x(t)$. We say that $x_{\rm SS}(t)$ is **observable**, because it is the solution visible in the graph after the **transients** (negative exponential terms) die out.

Readers may be mislead by the method of undetermined coefficients, in which it turns out that $x_p(t)$ and $x_{\rm SS}(t)$ are the same. Alternatively, a particular solution $x_p(t)$ can be calculated by variation of parameters, a method which produces in $x_p(t)$ extra terms containing negative exponentials. These extra terms come from the homogeneous solution – their appearance cannot always be avoided. This justifies the careful definition of steady–state solution, in which the transient terms are removed from $x_p(t)$ to produce $x_{\rm SS}(t)$.

Practical resonance is said to occur when the external frequency ω has been tuned to produce the largest possible steady–state amplitude.

Mathematically, this happens exactly when the amplitude function $C = C(\omega)$ defined in (8) has a maximum. If a maximum exists on $0 < \omega < \infty$, then $C'(\omega) = 0$ at the maximum. The derivative is computed by the power rule:

$$(9) \quad \begin{aligned} C'(\omega) &= \frac{-F_0}{2}\frac{2(k - m\omega^2)(-2m\omega) + 2c^2\omega}{((k - m\omega^2)^2 + (c\omega)^2)^{3/2}} \\ &= \omega\left(2mk - c^2 - 2m^2\omega^2\right)\frac{C(\omega)^3}{F_0^2} \end{aligned}$$

If $2km - c^2 \le 0$, then $C'(\omega)$ does not vanish for $0 < \omega < \infty$ and hence there is no maximum. If $2km - c^2 > 0$, then $2km - c^2 - 2m^2\omega^2 = 0$ has exactly one root $\omega = \sqrt{k/m - c^2/(2m^2)}$ in $0 < \omega < \infty$ and by $C(\infty) = 0$ it follows that $C(\omega)$ is a maximum. In summary:

> Practical resonance for $mx''(t) + cx'(t) + kx(t) = F_0 \cos \omega t$ occurs precisely when the external frequency ω is tuned to $\omega = \sqrt{k/m - c^2/(2m^2)}$ and $k/m - c^2/(2m^2) > 0$.

In Figure 22, the amplitude of the steady–state periodic solution is graphed against the external natural frequency ω, for the differential equation $x'' + cx' + 26x = 10 \cos \omega t$ and damping constants $c = 1, 2, 3$. The practical resonance condition is $\omega = \sqrt{26 - c^2/2}$. As c increases from 1 to 3, the maximum point $(\omega, C(\omega))$ satisfies a monotonicity condition: both ω and $C(\omega)$ decrease as c increases. The maxima for the three curves in the figure occur at $\omega = \sqrt{25.5}$, $\sqrt{24}$, $\sqrt{21.5}$. Pure resonance occurs when $c = 0$ and $\omega = \sqrt{26}$.

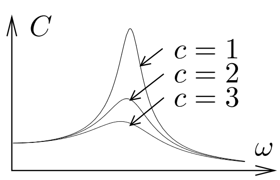

Figure 22. Practical resonance for $x'' + cx' + 26x = 10 \cos \omega t$: amplitude $C = 10/\sqrt{(26 - \omega^2)^2 + (c\omega)^2}$ versus external frequency ω for $c = 1, 2, 3$.

Uniqueness of the Steady–State Periodic Solution. Any two solutions of the nonhomogeneous differential equation (3) which are periodic of period $2\pi/\omega$ must be identical. The vehicle of proof is to show that their difference $x(t)$ is zero. The difference $x(t)$ is a solution of the homogeneous equation, it is $2\pi/\omega$–periodic and it has limit zero at infinity. A periodic function with limit zero must be zero, therefore the two solutions are identical. A more general statement is true:

> Consider the equation $mx''(t) + cx'(t) + kx(t) = f(t)$ with $f(t + T) = f(t)$ and m, c, k positive. Then a T–periodic solution is unique.

In Figure 23, the unique steady–state periodic solution is graphed for the differential equation $x'' + 2x' + 2x = \sin t + 2\cos t$. The transient solution of the homogeneous equation and the steady–state solution appear in Figure 24. In Figure 25, several solutions are shown for the differential equation $x'' + 2x' + 2x = \sin t + 2\cos t$, all of which reproduce eventually the steady–state solution $x = \sin t$.

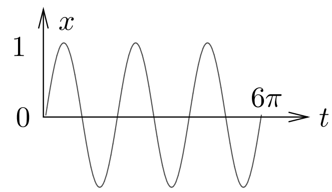

Figure 23. Steady-state periodic solution $x(t) = \sin t$ of the differential equation $x'' + 2x' + 2x = \sin t + 2\cos t$.

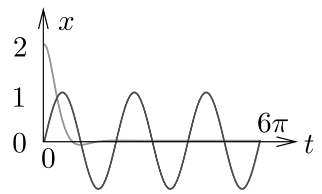

Figure 24. Transient solution of $x'' + 2x' + 2x = 0$ and the steady-state solution of $x'' + 2x' + 2x = \sin t + 2\cos t$.

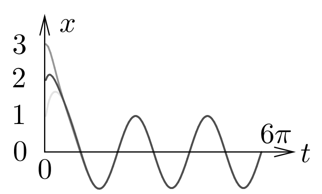

Figure 25. Solutions of $x'' + 2x' + 2x = \sin t + 2\cos t$ with $x'(0) = 1$ and $x(0) = 1, 2, 3$, all of which graphically coincide with the steady-state solution $x = \sin t$ for $t \geq \pi$.

Pseudo–Periodic Solution. Resonance gives rise to solutions of the form $x(t) = A(t)\sin(\omega t - \alpha)$ where $A(t)$ is a time–varying amplitude. Figure 26 shows such a solution, which is called a **pseudo–periodic solution** because it has a natural period $2\pi/\omega$ arising from the trigonometric factor $\sin(\omega t - \alpha)$. The only requirement on $A(t)$ is that it be non–vanishing, so that it acts like an amplitude. The **pseudo–period** of a pseudo–periodic solution can be determined graphically, by computing the length of time it takes for $x(t)$ to vanish three times.

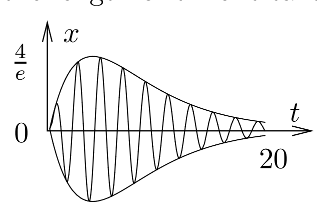

Figure 26. The pseudo-periodic solution $x = te^{-t/4}\sin(3t)$ of $16x'' + 8x' + 145x = 96e^{-t/4}\cos 3t$ and its envelope curves $x = \pm te^{-t/4}$.

Exercises 5.8

Resonance and Beats. Classify for resonance or beats. In the case of resonance, find an unbounded solution. In the case of beats, find the solution for $x(0) = 0$, $x'(0) = 0$, then graph it through a full period of the slowly-varying envelope.

1. $x'' + 4x = 10\sin 2t$

2. $x'' + 4x = 5\sin 2t$

3. $x'' + 100x = 10\sin 9t$

4. $x'' + 100x = 5\sin 9t$

5. $x'' + 25x = 5\sin 4t$

6. $x'' + 25x = 5\cos 4t$

7. $x'' + 16x = 5\sin 4t$

8. $x'' + 16x = 10\sin 4t$

Resonant Frequency and Amplitude. Consider $mx'' + cx' + kx = F_0\cos(\omega t)$. Compute the steady-state oscillation $A\cos(\omega t) + B\sin(\omega t)$, its amplitude $C = \sqrt{A^2 + B^2}$, the tuned practical resonance frequency ω^*, the resonant amplitude C^* and the ratio $100C/C^*$.

9. $m = 1$, $\beta = 2$, $k = 17$, $F_0 = 100$, $\omega = 4$.

10. $m = 1$, $\beta = 2$, $k = 10$, $F_0 = 100$, $\omega = 4$.

9. $m = 1$, $\beta = 4$, $k = 5$, $F_0 = 10$, $\omega = 2$.

10. $m = 1$, $\beta = 2$, $k = 6$, $F_0 = 10$, $\omega = 2$.

9. $m = 1$, $\beta = 4$, $k = 5$, $F_0 = 5$, $\omega = 2$.

10. $m = 1$, $\beta = 2$, $k = 5$, $F_0 = 5$, $\omega = 3/2$.