Definitions.

Pivot of A	A column in $\mathbf{rref}(A)$ which contains a leading one has a corresponding column in A, called a pivot column of A.
Basis of V	It is an independent set $\mathbf{v}_1, \ldots, \mathbf{v}_k$ from data set V whose linear combinations generate all data items in V. Generally, a basis is discovered by taking partial derivatives on symbols representing arbitrary constants.

Main Results.

Theorem 21 (Dimension)
If a vector space V has a basis $\mathbf{v}_1, \ldots, \mathbf{v}_p$ and also a basis $\mathbf{u}_1, \ldots, \mathbf{u}_q$, then $p = q$. The **dimension** of V is this unique number p.

Lemma 1 (Pivot Columns and Dependence) A non-pivot column of A is a linear combination of the pivot columns of A.

Theorem 22 (Independence)
The pivot columns of a matrix A are linearly independent.

Definitions.

$\mathbf{rank}(A)$	The number of leading ones in $\mathbf{rref}(A)$
$\mathbf{nullity}(A)$	The number of columns of A minus $\mathbf{rank}(A)$
Pivot of A	A column number in $\mathbf{rref}(A)$ which contains a leading one.

Main Results.

Theorem 23 (Rank-Nullity Equation)
$\mathbf{rank}(A) + \mathbf{nullity}(A) =$ column dimension of A

Theorem 24 (Row Rank Equals Column Rank)
The number of independent rows of a matrix A equals the number of independent columns of A. Equivalently, $\mathbf{rank}(A) = \mathbf{rank}(A^T)$.

Theorem 25 (Pivot Method)
Let A be the augmented matrix of $\mathbf{v}_1$, ..., $\mathbf{v}_k$. Let the leading ones in $\mathbf{rref}(A)$ occur in columns i_1, ..., i_p. Then a largest independent subset of the k vectors $\mathbf{v}_1$, ..., $\mathbf{v}_k$ is the set

$$\mathbf{v}_{i_1}, \mathbf{v}_{i_2}, \ldots, \mathbf{v}_{i_p}.$$

Definitions.

$\mathbf{kernel}(A) = \mathbf{nullspace}(A) = \{\mathbf{x} \ : \ A\mathbf{x} = \mathbf{0}\}$.

$\mathbf{Image}(A) = \mathbf{colspace}(A) = \{\mathbf{y} \ : \ \mathbf{y} = A\mathbf{x}$ for some $\mathbf{x}\}$.

$\mathbf{rowspace}(A) = \mathbf{colspace}(A^T) = \{\mathbf{w} \ : \ \mathbf{w} = A^T\mathbf{y}$ for some $\mathbf{y}\}$.

dim(V) is the number of elements in a basis for V.

How to Compute Null, Row, Column Spaces

Null Space. Compute $\mathbf{rref}(A)$. Write out the general solution $\mathbf{x}$ to $A\mathbf{x} = \mathbf{0}$, where the free variables are assigned parameter names t_1, ..., t_k. Report the basis for $\mathbf{nullspace}(A)$ as the list $\partial_{t_1}\mathbf{x}$, ..., $\partial_{t_k}\mathbf{x}$.

Column Space. Compute $\mathbf{rref}(A)$. Identify the pivot columns i_1, ..., i_k. Report the basis for $\mathbf{colspace}(A)$ as the list of columns i_1, ..., i_k of A.

Row Space. Compute $\mathbf{rref}(A^T)$. Identify the lead variable columns i_1, ..., i_k. Report the basis for $\mathbf{rowspace}(A)$ as the list of rows i_1, ..., i_k of A.

Alternatively, compute $\mathbf{rref}(A)$, then $\mathbf{rowspace}(A)$ has a (different) basis consisting of the list of nonzero rows of $\mathbf{rref}(A)$.

Theorem 26 (Dimension Identities)

(a) dim($\mathbf{nullspace}(A)$) = dim($\mathbf{kernel}(A)$) = $\mathbf{nullity}(A)$

(b) dim($\mathbf{colspace}(A)$) = dim($\mathbf{Image}(A)$) = $\mathbf{rank}(A)$

(c) dim($\mathbf{rowspace}(A)$) = $\mathbf{rank}(A)$

(d) dim($\mathbf{kernel}(A)$) + dim($\mathbf{Image}(A)$) = column dimension of A

(e) dim($\mathbf{kernel}(A)$) + dim($\mathbf{kernel}(A^T)$) = column dimension of A

An Equivalence Test in R^n

Assume given two sets of fixed vectors $\mathbf{v}_1, \ldots, \mathbf{v}_k$ and $\mathbf{u}_1, \ldots, \mathbf{u}_\ell$, in the same space R^n. A test will be developed for equivalence of bases, in a form suited for use in computer algebra systems and numerical laboratories.

Theorem 27 (Equivalence Test for Bases)
Define augmented matrices

$$\begin{aligned} B &= \mathbf{aug}(\mathbf{v}_1, \ldots, \mathbf{v}_k) \\ C &= \mathbf{aug}(\mathbf{u}_1, \ldots, \mathbf{u}_\ell) \\ W &= \mathbf{aug}(B, C) \end{aligned}$$

The relation

$$k = \ell = \mathbf{rank}(B) = \mathbf{rank}(C) = \mathbf{rank}(W)$$

implies

1. $\mathbf{v}_1, \ldots, \mathbf{v}_k$ is an independent set.

2. $\mathbf{u}_1, \ldots, \mathbf{u}_\ell$ is an independent set.

3. $\mathbf{span}\{\mathbf{v}_1, \ldots, \mathbf{v}_k\} = \mathbf{span}\{\mathbf{u}_1, \ldots, \mathbf{u}_\ell\}$

In particular, $\mathbf{colspace}(B) = \mathbf{colspace}(C)$ and each set of vectors is an equivalent basis for this vector space.

Proof: Because $\mathbf{rank}(B) = k$, then the first k columns of W are independent. If some column of C is independent of the columns of B, then W would have $k+1$ independent columns, which violates $k = \mathbf{rank}(W)$. Therefore, the columns of C are linear combinations of the columns of the columns of B. The vector space $U = \mathbf{colspace}(C)$ is therefore a subspace of the vector space $V = \mathbf{colspace}(B)$. Because each vector space has dimension k, then $U = V$. The proof is complete.

Equivalent Bases: Computer Illustration

The following `maple` code applies the theorem to verify that the two bases determined from the `colspace` command in `maple` and the pivot columns of A are equivalent. In `maple`, the report of the column space basis is identical to the nonzero rows of $\mathbf{rref}(A^T)$.

```
with(linalg):
A:=matrix([[1,0,3],[3,0,1],[4,0,0]]);
colspace(A);         # Solve Ax=0, basis v1,v2 below
v1:=vector([2,0,-1]);v2:=vector([0,2,3]);
rref(A);             # Find the pivot cols=1,3
u1:=col(A,1); u2:=col(A,3); # pivot col basis
B:=augment(v1,v2); C:=augment(u1,u2);
W:=augment(B,C);
rank(B),rank(C),rank(W); # Test requires all equal 2
```

Equivalent Bases

A false test. The relation

$$\mathbf{rref}(B) = \mathbf{rref}(C)$$

holds for a substantial number of examples. However, it does not imply that each column of C is a linear combination of the columns of B. For example, define

$$B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Then

$$\mathbf{rref}(B) = \mathbf{rref}(C) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix},$$

but $\mathbf{col}(C, 2)$ is not a linear combination of the columns of B. This means $V = \mathbf{colspace}(B)$ is not equal to $U = \mathbf{colspace}(C)$. Geometrically, V and U are planes in R^3 which intersect only along the line L through the two points $(0, 0, 0)$ and $(1, 0, 1)$.