Calculators, books, notes and computers are not allowed. Answer checks are not expected or required. First drafts are expected, not complete presentations. The midterm exam has 5 problems, some with multiple parts, suitable for 50 minutes.

1. (ch4) Complete enough of the following to add to 100%.

(a) [100%] Let $S$ be the vector space of all continuously differentiable functions defined on $-2 \leq x \leq 2$. Define $V$ to be the set of all functions $f(x)$ in $S$ such that $\int\limits_{-2}^{2} x f'(x) dx = 0$. Prove that $V$ is a subspace of $S$, by using the Subspace Criterion.

(b) [30%] Let $S$ be the set of all $3 \times 1$ column vectors $x$ with components $x_1$, $x_2$, $x_3$. Assume the usual $\mathbb{R}^3$ rules for addition and scalar multiplication. Let $V$ be the subset of $S$ defined by the dot product equations $(a + b) \cdot x = 0$, $b \cdot x = 0$, where $a$ and $b$ are vectors in $S$. Prove that $V$ is a subspace of $S$.

(c) [70%] Solve for the unknowns $a$, $b$, $c$, $d$ in the system of equations below by augmented matrix RREF methods, showing all details. Briefly, show the entire snapshot sequence to the $\text{rref}$, then display the general solution with variables $t_1$, $t_2$, . . . .

\[
\begin{align*}
a + b - 2c + d &= 2 \\
+ b + 2c + &= 0 \\
a + 2b + + d &= 2 \\
a + 3b + 2c + d &= 2
\end{align*}
\]

Solution 1(a). Use the subspace criterion: (a) Given $f$ and $g$ in $V$, write details to show $f + g$ is in $V$; (b) Given $f$ in $V$ and $k$ constant, write details to show $kf$ is in $V$. Let $h(x) = x$, which is a function in $S$. Details for (a): Given $\int\limits_{-2}^{2} f'(x)h(x) dx = 0$ and $\int\limits_{-2}^{2} g'(x)h(x) dx = 0$, add the equations to obtain the equation $\int\limits_{-2}^{2} (f'(x) + g'(x))h(x) dx = 0$. This finishes (a). Details for (b): Given $\int\limits_{-2}^{2} f'(x)h(x) dx = 0$ and $k$ constant, multiply the equation by $k$ and re-arrange factors to obtain the new equation $\int\limits_{-2}^{2} (kf'(x))h(x) dx = 0$. This proves (b).

Solution 1(b). Let $a$ and $b$ be given. Let $A$ be the matrix whose rows are $a + b$, $b$, $0$. Then the restriction equations given are equivalent to $Ax = 0$. By Theorem 2 in Edwards-Penney, $V$ is a subspace of $S$.

Solution 1(c). The answer is

\[
\begin{pmatrix}
2 + 4t_1 - t_2 \\
-2t_1 \\
t_1 \\
t_2
\end{pmatrix} = \begin{pmatrix}
2 \\
0 \\
0 \\
0
\end{pmatrix} + \begin{pmatrix}
4 \\
-2 \\
1 \\
0
\end{pmatrix} + \begin{pmatrix}
-1 \\
0 \\
0 \\
1
\end{pmatrix}.
\]

2. (ch5) Complete (a) and either (b) or (c). Do not do both (b) and (c).

(a) [30%] Given $4x''(t) + 4x'(t) + 5x(t) = 0$, which represents a damped spring-mass system with $m = 4$, $c = 4$, $k = 5$, solve the differential equation $[20\%]$ and classify the answer as over-damped, critically damped or under-damped $[10\%]$.

(b) [70%] Display by variation of parameters a particular solution $x_p$ for the equation $x'' + 2x' = f(t)$. Leave the answer in unevaluated integral form. Evaluate all symbols except $f(t)$ appearing in (33) of Edwards-Penney.

(c) [70%] Find by undetermined coefficients the steady-state periodic solution for the equation $x'' + 2x' + 2x = 5\sin(t)$.

Solution 2(a). Use $4r^2 + 4r + 5 = 0$ and the quadratic formula to obtain roots $r_1 = -1/2 + i$, $r_2 = -1/2 - i$. Case 3 of the recipe gives $x(t) = c_1 e^{-t/2} \cos t + c_2 e^{-t/2} \sin t$. This is under-damped.
Solution 2(b).  
Solve \( x'' + 2x' = 0 \) by the recipe to get \( x_h = c_1x_1 + c_2x_2, \ x_1 = 1, \ x_2 = e^{-2t} \). Compute the Wronskian \( W = x_1x_2' - x_1'x_2 = -2e^{-2t} \). Then
\[
x_p = x_1 \int x_2 \frac{-f}{W} \, dt + x_2 \int x_1 \frac{f}{W} \, dt
\]
becomes
\[
x_p = \int \frac{f}{2} \, dt + e^{-2t} \int \frac{-f(t)e^{2t}}{2} \, dt.
\]

Solution 2(c). The trial solution is \( x = d_1 \cos t + d_2 \sin t \). Substitute the trial solution to obtain the answers \( d_1 = -2, \ d_2 = 1 \). The unique periodic solution \( x_{ss} \) is extracted from the general solution \( x = x_h + x_p \) by crossing out all negative exponential terms (terms which limit to zero at infinity). If \( x = x_h + x_p \) and \( x_p = d_1 \cos t + d_2 \sin t = -2 \cos t + \sin t \), then all terms of \( x_h = c_1e^{-t} \cos t + c_2e^{-t} \sin t \) are crossed out, giving the steady-state solution
\[
x_{ss} = -2 \cos t + \sin t.
\]

3. (ch5) Complete all parts below.

(a) [75%] Determine for \( y'' - 9y'' = 2xe^{3x} + 3x^3 + 2 \cos 3x + \sin 3x \) the corrected trial solution for \( y_p \) according to the method of undetermined coefficients. To save time, do not evaluate the undetermined coefficients (that is, do undetermined coefficient steps 1 and 2, but skip steps 3 and 4)!

Undocumented detail or guessing earns no credit.

(b) [25%] Using the recipe for higher order constant-coefficient differential equations, write out the general solution when the characteristic equation is \( (r^2 - 4)^3(r + 2)(r^2 + 6r + 10)^2 = 0 \).

Solution 3(a).
The homogeneous solution is \( y_h = c_1 + c_2x + c_3x^2 + c_4e^{2x} + c_5e^{-2x} \), because the characteristic polynomial has roots 0, 0, 0, 2, −2.

1. An initial trial solution \( y \) is constructed for atoms 1, \( x, x^2, x^3, e^{3x}, xe^{3x}, \cos 3x, \sin 3x \) giving
\[
y = y_1 + y_2 + y_3,
\]
\[
y_1 = d_1 + d_2x + d_3x^2 + d_4x^3,
\]
\[
y_2 = (d_5 + d_6x)e^{3x},
\]
\[
y_3 = d_7 \cos 3x + d_8 \sin 3x.
\]

Linear combinations of the listed independent atoms are supposed to reproduce, by assignment of constants, all derivatives of the right side of the differential equation.

2. The fixup rule \([d_j \rightarrow d_s x^s]\) is applied individually to each atom to give the corrected trial solution
\[
y = y_1 + y_2 + y_3,
\]
\[
y_1 = x^3(d_1 + d_2x + d_3x^2 + d_4x^3),
\]
\[
y_2 = x(d_5 + d_6x)e^{3x},
\]
\[
y_3 = d_7 \cos 3x + d_8 \sin 3x.
\]

The powers \( x^s \) multiplied in each case are designed to eliminate terms in the initial trial solution which duplicate atoms appearing in the homogeneous solution \( y_h \). The factor is exactly \( x^s \) of the Edwards-Penney table, where \( s \) is the multiplicity of the characteristic equation root \( r \) that produced the related atom in the homogeneous solution \( y_h \). By design, unrelated atoms are unaffected by the fixup rule \( [s_j = 0 \text{ in this case and factor } x^0 = 1] \) and that is why \( y_3 \) was unaltered.

3. Undetermined coefficient step skipped, according to the problem statement.

4. Undetermined coefficient step skipped, according to the problem statement.

Solution 3(c).
Write \( (r^2 - 4)^3(r + 2)(r^2 + 6r + 10)^2 = 0 \) as \( (r - 2)^3(r + 2) \frac{1}{4}(r^2 + 6r + 10)^2 = 0 \), then \( y = u_1e^{2x} + u_2e^{-2x} + u_3e^{-3x} \cos x + u_4e^{-3x} \sin x \). The polynomials are \( u_1 = c_1 + c_2x + c_3x^2 \) (3 terms for multiplicity 3), \( u_2 = c_4 + c_5x + c_6x^2 + c_7x^3 \) (4 terms for multiplicity 4), \( u_3 = c_8 + c_9x, u_4 = c_{10} + c_{11}x \).
4. (ch6) Complete all of the items below.

(a) [40%] Find the eigenvalues of the matrix \( A = \begin{bmatrix} 4 & 1 & -1 & 0 \\ 0 & 5 & -2 & 1 \\ 0 & 0 & 1 & -4 \\ 0 & 0 & 5 & 1 \end{bmatrix} \). To save time, do not find eigenvectors!

(b) [60%] Given \( A = \begin{bmatrix} 1 & 1 & -1 \\ 0 & 5 & 1 \\ 0 & 1 & 5 \end{bmatrix} \), then there exists an invertible matrix \( P \) and a diagonal matrix \( D \) such that \( AP = PD \). Find a column of \( P \) so that 6 appears in the same column of \( D \).

Solution 4(a).
Subtract \( \lambda \) from the diagonal elements of \( A \) and expand the determinant \( \text{det}(A - \lambda I) \) to obtain the characteristic polynomial \((4 - \lambda)(5 - \lambda)[(1 - \lambda)(1 - \lambda) + 20] = 0\). The eigenvalues are the roots: \( \lambda = 4, 5, 1 + 2\sqrt{5}i, 1 - 2\sqrt{5}i \).

Used here was the cofactor rule for determinants. Sarrus’ rule does not apply for \( 4 \times 4 \) determinants (an error) and the triangular rule likewise does not directly apply (another error).

Solution 4(b).
Details: According to the theory of diagonalizable matrices, \( P \) is the matrix package of eigenvectors and \( D \) is the matrix package of eigenvalues. There are \( 3! = 6 \) possible orderings to make these packages, hence 6 possible choices exist, all of which are correct. In all cases, an eigenelement is entered in the same location in each package.

One way to package:
\[
P = \begin{pmatrix} 0 & 1 & 2 \\ 1 & 0 & 3 \\ 1 & 0 & -3 \end{pmatrix}, \quad D = \begin{pmatrix} 6 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{pmatrix}.
\]

The problem is solved without displaying the packages. It suffices to recognize that \( Ax = 6x \) defines the eigenvector \( x \) corresponding to \( \lambda = 6 \). Solving gives column 1 of \( P \) above.

5. (ch6) Complete all parts below.

Consider a given \( 3 \times 3 \) matrix \( A \) having three eigenpairs

\[
5, \begin{pmatrix} 4 \\ 5 \\ 0 \end{pmatrix}; \quad -3, \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}; \quad 3, \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix}.
\]

(a) [50%] Display the vector general solution \( x(t) \) of the linear differential system \( x' = Ax \).

(b) [20%] Write a matrix algebra formula for the matrix \( A \) of (a) above. To save time, do not evaluate anything.

(c) [30%] Let \( B \) be a certain \( 2 \times 2 \) matrix. Fourier’s model for the computation of \( Bx \) is known to be

\[
x = c_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} + c_2 \begin{pmatrix} -1 \\ 2 \end{pmatrix} \quad \text{implies} \quad Bx = -c_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} - 3c_2 \begin{pmatrix} -1 \\ 2 \end{pmatrix}.
\]

Find \( B \).

Solution 5(a).
Answer: The eigenanalysis method implies

\[
x(t) = c_1 e^{5t} \begin{pmatrix} 4 \\ 5 \\ 0 \end{pmatrix} + c_2 e^{-3t} \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix} + c_3 e^{3t} \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix}.
\]
Solution 5(b).

\[ AP = PD \] implies \( A = PDP^{-1} \).

Solution 5(c).

Answer: Fourier’s model implies the eigenpairs of \( B \) are given by

\[
\left( -1, \left( \begin{array}{c} 1 \\ 2 \end{array} \right) \right), \quad \left( -3, \left( \begin{array}{c} -1 \\ 2 \end{array} \right) \right).
\]

Then \( D = \left( \begin{array}{cc} -1 & 0 \\ 0 & -3 \end{array} \right) \) and \( P = \left( \begin{array}{cc} 1 & -1 \\ 2 & 2 \end{array} \right) \) implies by \( BP = PD \) that \( B = PDP^{-1} \) and finally

\[
B = \left( \begin{array}{cc} -2 & 1/2 \\ 2 & -2 \end{array} \right).
\]