

1. (ch4) Complete enough of the following to add to 100%.

(a) [100%] Let  $S$  be the vector space of all continuous functions defined on  $-2 \leq x \leq 2$ . Define  $V$  to be the set of all functions  $f(x)$  in  $S$  such that  $\int_0^2 xf(x)dx = 0$ . Prove that  $V$  is a subspace of  $S$ , by using the Subspace Criterion.

(b) [30%] Let  $S$  be the set of all  $3 \times 1$  column vectors  $\mathbf{x}$  with components  $x_1, x_2, x_3$ . Assume the usual  $\mathcal{R}^3$  rules for addition and scalar multiplication. Let  $V$  be the subset of  $S$  defined by the dot product equations  $\mathbf{a} \cdot \mathbf{x} = 0$ ,  $\mathbf{b} \cdot \mathbf{x} = 0$ , where  $\mathbf{a}$  and  $\mathbf{b}$  are vectors in  $S$ . Prove that  $V$  is a subspace of  $S$ .

(c) [70%] Solve for the unknowns  $a, b, c, d$  in the system of equations below by augmented matrix RREF methods, showing all details. Briefly, show the entire snapshot sequence to the **rref**, then display the general solution with variables  $t_1, t_2, \dots$ .

$$\begin{aligned} a + b - 2c + d &= 1 \\ + b + 2c + &= 0 \\ a + 2b + &+ d = 1 \\ a + 3b + 2c + d &= 1 \end{aligned}$$

① Zero is in  $V$  because  $\int_0^2 0 dx = 0$ . If  $f, g$  are in  $V$ , then  $\int_0^2 (c_1 f + c_2 g) x dx = c_1 (\int_0^2 x f dx) + c_2 (\int_0^2 x g dx) = c_1(0) + c_2(0) = 0$ . By the subspace criterion,  $V$  is a subspace of  $S$ .

② Let matrix  $A$  have rows  $\vec{a}, \vec{b}, \vec{0}$ , that is,  $A = \text{aug}(\vec{a}, \vec{b}, \vec{0})^T$ . Then the dot product equations are exactly  $A\vec{x} = \vec{0}$ . By Theorem 2 of E&P,  $V$  is a subspace of  $S$ .

③  $\left( \begin{array}{ccc|c|c} 1 & 1 & -2 & 1 & 1 \\ 0 & 1 & 2 & 0 & 0 \\ 1 & 2 & 0 & 1 & 1 \\ 1 & 3 & 2 & 1 & 1 \end{array} \right)$

$$\begin{cases} a - 4c + d = 1 \\ b + 2c = 0 \end{cases}$$

$$\left( \begin{array}{ccc|c|c} 1 & 1 & -2 & 1 & 1 \\ 0 & 1 & 2 & 0 & 0 \\ 0 & 1 & 2 & 0 & 0 \\ 0 & 2 & 4 & 0 & 0 \end{array} \right) \begin{array}{l} \text{combo} \\ \text{combo} \end{array}$$

$$\left( \begin{array}{ccc|c|c} 1 & 1 & -2 & 1 & 1 \\ 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right) \begin{array}{l} \text{Combo} \\ \text{Combo} \end{array}$$

$$\left( \begin{array}{ccc|c|c} 1 & 0 & -4 & 1 & 1 \\ 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right) \begin{array}{l} \text{Combo} \\ \text{rref} \\ \text{found} \end{array}$$

gen  $\begin{cases} a = 4t_1 - t_2 + 1 \\ b = -2t_1 \\ c = t_1 \\ d = t_2 \end{cases}$

Sol  $\begin{cases} a = 4t_1 - t_2 + 1 \\ b = -2t_1 \\ c = t_1 \\ d = t_2 \end{cases}$

$$\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = t_1 \begin{pmatrix} 4 \\ -2 \\ 1 \\ 0 \end{pmatrix} + t_2 \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

2. (ch5) Complete (a) and either (b) or (c). Do not do both (b) and (c).

(a) [30%] Given  $8x''(t) + 16x'(t) + 2x(t) = 0$ , which represents a damped spring-mass system with  $m = 8$ ,  $c = 16$ ,  $k = 2$ , solve the differential equation [20%] and classify the answer as over-damped, critically damped or under-damped [10%].

(b) [70%] Display by variation of parameters a particular solution  $x_p$  for the equation  $x'' + 2x' = f(t)$ . Leave the answer in unevaluated integral form. Evaluate all symbols except  $f(t)$  appearing in (33) of Edwards-Penney.

(c) [70%] Find by undetermined coefficients the steady-state periodic solution for the equation  $x'' + 2x' + 2x = 5 \sin(t)$ .

a

$$4r^2 + 8r + 1 = 0$$

$$r = \frac{-8 \pm \sqrt{64 - 16}}{8}$$

$$= -1 \pm \frac{1}{2}\sqrt{3}$$

$$= r_1, r_2$$

$$x(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}$$

over-damped

$$r_1 = -1 + \sqrt{3}/2, r_2 = -1 - \sqrt{3}/2$$

b

$$r^2 + 2r = 0$$

$$r(r+2) = 0$$

$$r = 0, -2$$

$$x_1 = e^{0t} = 1$$

$$x_2 = e^{-2t}$$

$$W = \begin{vmatrix} 1 & e^{-2t} \\ 0 & -2e^{-2t} \end{vmatrix}$$

$$= -2e^{-2t}$$

$$(33) x_p = \left( \int x_2 \frac{(-f)}{W} \right) x_1 + \left( \int x_1 \frac{f}{W} \right) x_2$$

$$x_p = \left( \int \frac{f(t)}{2} dt \right) + \left( \int -\frac{f(t)}{2} e^{2t} dt \right) e^{-2t}$$

c

$$x = d_1 \cos t + d_2 \sin t$$

$$x' = -d_1 \sin t + d_2 \cos t$$

$$x'' = -d_1 \cos t - d_2 \sin t$$

$$x'' + 2x' + 2x = 5 \sin t$$

$$2x' + x = 5 \sin t$$

because  $x'' + x = 0$

$$-2d_1 \sin t + 2d_2 \cos t + d_1 \cos t + d_2 \sin t = 5 \sin t$$

$$\begin{cases} -2d_1 + d_2 = 5 \\ 2d_2 + d_1 = 0 \end{cases}$$

$$d_1 = -2, d_2 = 1$$

$$\lim_{t \rightarrow \infty} x_h(t) = 0 \Rightarrow$$

match atoms left and right

$$\text{periodic solution equals}$$

$$x(t) = -2 \cos t + \sin t$$

3. (ch5) Complete all parts below.

(a) [75%] Determine for  $y'' - 9y''' = xe^{3x} + x^3 + 2\sin 3x$  the **corrected** trial solution for  $y_p$  according to the method of undetermined coefficients. To save time, **do not** evaluate the undetermined coefficients (that is, do undetermined coefficient steps **1** and **2**, but skip steps **3** and **4**)! Undocumented detail or guessing earns no credit.

(b) [25%] Using the *recipe* for higher order constant-coefficient differential equations, write out the general solution when the characteristic equation is  $(r^2 - 1)^3(r + 1)(r^2 + 6r + 10)^2 = 0$ .

(a) atom list of RHS =  $1, x, x^2, x^3, e^{3x}, xe^{3x}, \cos 3x, \sin 3x$

$$\text{char eq } r^5 - 9r^3 = 0$$

$$\text{roots} = 0, 0, 0, 3, -3$$

$$y = (d_1 + d_2 x + d_3 x^2 + d_4 x^3) x^3 + [(d_5 + d_6 x) e^{3x}] x + d_7 \cos 3x + d_8 \sin 3x$$

$s=3$  for multiplier  $x^s$  on coefficients  $d_1 \rightarrow d_4$

$s=1$  for multiplier  $x^s$  on coefficients  $d_5, d_6$

$s=0$  for  $d_7, d_8$ .

(b)  $(r-1)^3(r+1)^4((r+3)^2+1)^2 = 0$

$$\begin{cases} y = u_1 e^x + u_2 e^{-x} + (u_3 \cos x + u_4 \sin x) e^{-3x} \\ u_1 = c_1 + c_2 x + c_3 x^2 \\ u_2 = c_4 + c_5 x + c_6 x^2 + c_7 x^3 \\ u_3 = c_8 + c_9 x \\ u_4 = c_{10} + c_{11} x \end{cases}$$

11 constants because of 11 roots.

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4. (ch6) Complete all of the items below.

(a) [40%] Find the eigenvalues of the matrix  $A = \begin{bmatrix} 4 & 1 & -1 & 0 \\ 0 & 5 & -2 & 1 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 3 & 1 \end{bmatrix}$ . To save time, **do not** find eigenvectors!

(b) [60%] Given  $A = \begin{bmatrix} 1 & 1 & -1 \\ 0 & 5 & 1 \\ 0 & 1 & 5 \end{bmatrix}$ , then there exists an invertible matrix  $P$  and a diagonal matrix  $D$  such that  $AP = PD$ . Find a column of  $P$  so that 4 appears in the same column of  $D$ .

$$(a) \begin{vmatrix} 4-\lambda & 1 & -1 & 0 \\ 0 & 5-\lambda & -2 & 1 \\ 0 & 0 & 1-\lambda & -3 \\ 0 & 0 & 3 & 1-\lambda \end{vmatrix} = 0$$

$$(4-\lambda)(5-\lambda)((1-\lambda)^2 + 9) = 0 \quad \text{by cofactor expansion}$$

$$\boxed{\lambda = 4, 5, 1+3i, 1-3i}$$

(b) Solve  $A\vec{x} = 4\vec{x}$  for  $\vec{x} \neq \vec{0}$ :

$$\left( \begin{array}{ccc|c} -3 & 1 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & 1 & 0 \end{array} \right)$$

$$\left( \begin{array}{ccc|c} 3 & -1 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right) \quad \begin{array}{l} \text{mult} \\ \text{combo} \end{array}$$

$$\left( \begin{array}{ccc|c} 3 & 0 & 2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right) \quad \text{rref found}$$

$$\begin{cases} x_1 + \frac{2}{3}x_3 = 0 \\ x_2 + x_3 = 0 \end{cases}$$

gen sol

$$\begin{cases} x_1 = -2t_1/3 \\ x_2 = -t_1 \\ x_3 = t_1 \end{cases}$$

$$\text{col} = \begin{pmatrix} -2/3 \\ -1 \\ 1 \end{pmatrix}$$

or any scalar multiple of this answer

Use this page to start your solution. Staple extra pages as needed.

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5. (ch6) Complete all parts below.

Consider a given  $3 \times 3$  matrix  $A$  having three eigenpairs

$$5, \begin{pmatrix} 2 \\ 3 \\ 0 \end{pmatrix}; \quad -3, \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}; \quad 3, \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix}.$$

(a) [50%] Display the vector general solution  $\mathbf{x}(t)$  of the linear differential system  $\mathbf{x}' = A\mathbf{x}$ .

(b) [20%] Write a matrix algebra formula for the matrix  $A$  of (a) above. To save time, do not evaluate anything.

(c) [30%] Let  $B$  be a certain  $2 \times 2$  matrix. Fourier's model for the computation of  $B\mathbf{x}$  is known to be

typo corrected in class

$$\mathbf{x} = c_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} + c_2 \begin{pmatrix} -1 \\ 2 \end{pmatrix} \quad \text{implies}$$

$$\textcircled{B} \textcircled{A}\mathbf{x} = -c_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} - 3c_2 \begin{pmatrix} -1 \\ 2 \end{pmatrix}.$$

Find  $B$ .

$$\textcircled{a} \quad \vec{x}(t) = c_1 e^{5t} \begin{pmatrix} 2 \\ 3 \\ 0 \end{pmatrix} + c_2 e^{-3t} \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix} + c_3 e^{3t} \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix}$$

$$\textcircled{b} \quad AP = PD \quad P = \begin{pmatrix} 2 & -1 & 2 \\ 3 & 2 & -1 \\ 0 & 1 & 0 \end{pmatrix} \quad D = \begin{pmatrix} 5 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

$$A = PDP^{-1}$$

$$\textcircled{c} \quad P = \begin{pmatrix} 1 & -1 \\ 2 & 2 \end{pmatrix} \quad D = \begin{pmatrix} -1 & 0 \\ 0 & -3 \end{pmatrix}$$

$$BP = PD \Rightarrow B = PDP^{-1}$$

$$= \begin{pmatrix} 1 & -1 \\ 2 & 2 \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & -3 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ -2 & 1 \end{pmatrix} \cdot \frac{1}{4}$$

$$= \begin{pmatrix} -1 & 3 \\ -2 & -6 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ -2 & 1 \end{pmatrix} \frac{1}{4}$$

$$= \begin{pmatrix} -8 & 2 \\ 8 & -8 \end{pmatrix} \frac{1}{4}$$

$$= \boxed{\begin{pmatrix} -2 & 1/2 \\ 2 & -2 \end{pmatrix}}$$

Use this page to start your solution. Staple extra pages as needed.