Applied Differential Equations 2250

Sample Midterm Exam 2, 7:30 and 10:45

Exam date: Tuesday, 28 March 2006

Instructions: This in-class exam is 50 minutes. No calculators, notes, tables or books. No answer check is expected. Details count 75%. The answer counts 25%.

1. (rref)

(a) Determine b such that the system has infinitely many solutions:

$$\begin{array}{rcrcrcl} x &+& 2y &+& z &=& b \\ 3x &+& y &+& 2z &=& 2b \\ 4x &+& 3y &+& 3z &=& 1+b \end{array}$$

Answer check (a) in maple:

```
A:=matrix([[1,2,1,b],[3,1,2,2*b],[4,3,3,1+b]]);
 A1:=addrow(A,1,3,-1);
 A2:=addrow(A1,2,3,-1);
1-2b == 0 gives infinitely many solutions
```

(b) Determine a, b such that the system has infinitely many solutions:

$$\begin{array}{rcrcrcl} x &+& 2y &+& z &=& a \\ 5x &+& y &+& 2z &=& 3a \\ 6x &+& 3y &+& bz &=& 1+a \end{array}$$

Answer check (b) in maple:

```
with(linalg):
A:=matrix([[1,2,1,a],[5,1,2,3*a],[6,3,b,1+a]]);
A1:=addrow(A,1,3,-1);
A2:=addrow(A1,2,3,-1);
A3:=addrow(A2,1,2,-5);
A4:=mulrow(A3,2,-1/9);
A5:=addrow(A4,2,1,-2);
-3+b == 0 and 3a-1 == 0 gives one free variable and infinitely many solutions
```

2. (vector spaces)

(a) Give two examples of vector spaces of functions, one of dimension two and one of dimension three.
(b) Let S be the vector space of all continuous functions on the real line and let V be the subset of S given by all solutions of the differential equation $y' = -2y$. Prove that V is a subspace of S.
(c) Find a basis for the subspace of $\mathcal{R}^3$ given by the system of equations

$$\begin{aligned} x + 2y - z &= 0, \\ x + y - 2z &= 0, \\ y + z &= 0, \end{aligned}$$

Answers:
(a) $V = \{c_1 + c_2 t\}$ and $W = \{c_1 + c_2 t + c_3 t^2\}$ are vector spaces of polynomials with $\dim(V) = 2$ and $\dim(W) = 3$.
(b) All functions y in V look like $y = c_1 e^{-2t}$. Adding two such functions gives a function in V and

multiplying such a function by a scalar gives a function in V. Then V is closed under addition and scalar multiplication. Therefore, V is a subspace of S, by the subspace criterion.
(c) The general solution is

$$x = 3t_1, \quad y = -t_1, \quad z = t_1.$$

A basis is $\begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix}$.

3. (independence)

(a) Let $\mathbf{u} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$, $\mathbf{v} = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$, $\mathbf{w} = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$. State and apply a test that shows $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ are dependent.
(b) Extract from the list below a largest set of independent vectors.

$$\mathbf{a} = \begin{pmatrix} 1 \\ -1 \\ 0 \\ -1 \end{pmatrix}, \mathbf{b} = \begin{pmatrix} 2 \\ -2 \\ 0 \\ -2 \end{pmatrix}, \mathbf{c} = \begin{pmatrix} 3 \\ -1 \\ 0 \\ 1 \end{pmatrix}, \mathbf{d} = \begin{pmatrix} 0 \\ 2 \\ 0 \\ 4 \end{pmatrix}, \mathbf{e} = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 3 \end{pmatrix}.$$

(c) Assume that matrix D is invertible. Prove that $D\mathbf{a}$, $D\mathbf{b}$, $D\mathbf{c}$ are independent if and only if $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ are independent.

Answer checks and details:
(a) The test says that three vectors are dependent if and only if the rref of the augmented matrix of the three vectors has rank different from three.
Let `A:=matrix([[1,2,1],[-1,1,2],[1,0,-1]]);` and compute `rref(A);` in maple. Then the rref has a row of zeros, so the vectors are dependent.
(b) Let `A:=matrix([[1,2,3,0,1],[-1,-2,-1,2,1],[0,0,0,0,0],[-1,-2,1,4,3]]);`
and compute `rref(A);` in maple. The position of leading ones identifies $\mathbf{a}$, $\mathbf{c}$ as independent.
(c) A linear combination of $D\mathbf{a}$, $D\mathbf{b}$, $D\mathbf{c}$ equals zero if and only if D times the same linear combination of $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ equals zero. Since D is invertible, $D(c_1\mathbf{a} + c_2\mathbf{b} + c_3\mathbf{c}) = \mathbf{0}$ if and only if $c_1\mathbf{a} + c_2\mathbf{b} + c_3\mathbf{c} = \mathbf{0}$, proving independence of one set is equivalent to independence of the other set.

4. (determinants and elementary matrices)

(a)Assume given 3×3 matrices A, B. Suppose $B = E_1E_2A$ and E_1, E_2 are elementary matrices representing swap rules. Explain precisely why $\det(B) = \det(A)$.
(b) Let A and B be two 7×7 matrices such that AB contains two duplicate rows. Explain precisely why either $\det(A)$ or $\det(B)$ is zero.

Answers and details:
(a) $\det(B)$=$\det(E1)\det(E2)\det(A)$ by the product rule for determinants. Each swap rule has determinant -1. So $\det(B) = (-1)(-1)\det(A)$ =$\det(A)$.
(b) $\det(AB) = 0$ because the determinant has two duplicate rows. Then $\det(A)\det(B) = 0$ by the product theorem for determinants. Hence either $\det(A) = 0$ or $\det(B) = 0$.

5. (inverses and Cramer's rule)

(a) Determine all values of x for which A^{-1} exists: $A = \begin{pmatrix} 1 & 2 & 0 \\ 2 & 0 & -3 \\ 0 & x & 1 \end{pmatrix}$.

(b) Solve for y in $A\mathbf{u} = \mathbf{b}$ by Cramer's rule: $A = \begin{pmatrix} 1 & 2 & 0 \\ 3 & 0 & 2 \\ 2 & -2 & 1 \end{pmatrix}$, $\mathbf{u} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$.

Answer checks in maple:
(a) Let `A:=matrix([[1,2,0],[2,0,-3],[0,x,1]]);` then compute `det(A);` to give $x \neq 4/3$, because the unique solution case is exactly $\det(A) \neq 0$.
(b) Let `A:=matrix([[1,2,0],[3,0,2],[2,-2,1]]);` and `b:=vector([1,0,-1]);` then compute from `linsolve(A,b);` to conclude that $x = 0$, $y = 1/2$, $z = 0$.
The hand solution uses Cramer's rule to give $y = \Delta_2/\Delta$ where $\Delta =$det(A) and Δ_1 is the same determinat but column 2 replaced by vector $\mathbf{b}$.