Applied Differential Equations 2250
Sample Midterm Exam 2, 7:30 and 10:45
Monday, 31 October 2005

Instructions: This in-class exam is 50 minutes. No calculators, notes, tables or books. No answer check is expected. Details count 75%. The answer counts 25%.

1. (rref)
(a) Determine b such that the system has infinitely many solutions:

$$\begin{array}{rcrcrcl} x & + & 2y & + & z & = & b \\ 3x & + & y & + & 2z & = & 2b \\ 4x & + & 3y & + & 3z & = & 1+b \end{array}$$

Answer: $1 - 2b = 0$

(b) Determine a, b such that the system has infinitely many solutions:

$$\begin{array}{rcrcrcl} x & + & 2y & + & z & = & a \\ 5x & + & y & + & 2z & = & 3a \\ 6x & + & 3y & + & bz & = & 1+a \end{array}$$

Answer: $-3 + b = 0$ and $3a - 1 = 0$

2. (vector spaces)
(a) Give two examples of vector spaces of functions, one of dimension two and one of dimension three.
(b) Let V be the vector space of all continuous functions on the real line and let S be the subset of V given by all solutions of the differential equation $y' = -2y$. Prove that S is a subspace of V.
(c) Find a basis for the subspace of $\mathcal{R}^3$ given by the system of equations

$$\begin{aligned} x + 2y - z &= 0, \\ x + y - 2z &= 0, \\ y + z &= 0, \end{aligned}$$

Answer (a): $V = \{c_1 + c_2 t\}$ and $W = \{c_1 + c_2 t + c_3 t^2\}$ are vector spaces of polynomials with $\dim(V) = 2$ and $\dim(W) = 3$.
Solution (b): All functions y in Slook like $y = c_1 e^{-2t}$. Adding two such functions gives a function in S and multiplying such a function by a scalar gives a function in S. Then S is closed under addition and scalar multiplication. Therefore, S is a subspace of V.

Answer (c): $x = 3t_1$, $y = -t_1$, $z = t_1$. A basis is $\partial_{t_1}\mathbf{x} = \begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix}$.

3. (independence)
(a) Let $\mathbf{u} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$, $\mathbf{v} = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$, $\mathbf{w} = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$. State and apply a test that shows $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ are dependent.
(b) Extract from the list below a largest set of independent vectors.

$$\mathbf{a} = \begin{pmatrix} 1 \\ -1 \\ 0 \\ -1 \end{pmatrix}, \mathbf{b} = \begin{pmatrix} 2 \\ -2 \\ 0 \\ -2 \end{pmatrix}, \mathbf{c} = \begin{pmatrix} 3 \\ -1 \\ 0 \\ 1 \end{pmatrix}, \mathbf{d} = \begin{pmatrix} 0 \\ 2 \\ 0 \\ 4 \end{pmatrix}, \mathbf{e} = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 3 \end{pmatrix}.$$

Solution (a): Let $A = \mathbf{aug}(\mathbf{u}, \mathbf{v}, \mathbf{w})$. The test is

rref(A) *has three leading ones if and only if* $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ *are independent.*

Computing **rref**(A) shows it has a row of zeros, so the vectors are dependent.
Solution (b): Let $A = \mathbf{aug}(\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}, \mathbf{e})$. Find **rref**$(A)$. Then leading ones are in columns 1 and 3, giving corresponding columns $\mathbf{a}$, $\mathbf{c}$ as the pivot columns of A. Answer: $\{\mathbf{a}, \mathbf{c}\}$ is a largest independent subset.

4. (determinants and elementary matrices)
(a)Assume given 3×3 matrices A, B. Suppose $B = E_1 E_2 A$ and E_1, E_2 are elementary matrices representing swap rules. Explain precisely why $\det(B) = \det(A)$.
(b) Let A and B be two 7×7 matrices such that AB contains two duplicate rows. Explain precisely why either $\det(A)$ or $\det(B)$ is zero.

Solution (a): $\det(B) = \det(E_1)\det(E_2)\det(A)$ by the product rule for determinants. Each swap rule has determinant -1. So $\det(B) = (-1)(-1)\det(A) = \det(A)$.
Solution (b): $\det(AB) = 0$ because the determinant has two duplicate rows. Then $\det(A)\det(B) = 0$ by the product theorem for determinants. Hence either $\det(A) = 0$ or $\det(B) = 0$.

5. (inverses and Cramer's rule)

(a) Determine all values of x for which A^{-1} exists: $A = \begin{pmatrix} 1 & 2 & 0 \\ 2 & 0 & -3 \\ 0 & x & 1 \end{pmatrix}$.

(b) Solve for y in $A\mathbf{u} = \mathbf{b}$ by Cramer's rule: $A = \begin{pmatrix} 1 & 2 & 0 \\ 3 & 0 & 2 \\ 2 & -2 & 1 \end{pmatrix}$, $\mathbf{u} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$.

Answer (a): $\det(A) \neq 0$ which is $x \neq 4/3$.
Answer (b): $\Delta = 6$, $x = 0$, $y = 1/2$, $z = 0$. The answer for y only is obtained as a quotient of two determinants, so it takes less time than finding all three values.