1. (ref)
Determine \(a, b\) such that the system has (1) infinitely many solutions, (2) no solutions.

\[
\begin{align*}
x + 6y + z &= 1 + a \\
5x + 3y + 2z &= 3 + 3a \\
6x + 9y + 3bz &= 2 + a \\
\end{align*}
\]

\[
\begin{array}{cc|c}
1 & 6 & 1 + a \\
5 & 3 & 3 + 3a \\
6 & 9 & 2 + a \\
\end{array}
\]

\[
\begin{array}{cc|c}
1 & 6 & 1 + a \\
0 & -27 - 3 & -2 - 2a \\
0 & -27 & -4 - 5a \\
\end{array}
\]

\[
\begin{array}{cc|c}
1 & 6 & 1 + a \\
0 & 27 & 2 + 2a \\
0 & 0 & -2 - 3a \\
\end{array}
\]

Answer (1): \(\infty\) - many sols \(\iff\) last row is all zeros
\(\iff\) \(b = 1, a = -2/3\)

Answer (2): No solutions \(\iff\) singular equation
\(\iff\) last row is 0 0 0 \(x\) with \(x \neq 0\)
\(\iff\) \(b = 1, a \neq -2/3\)
2. (vector spaces)
   (a) [25%] Give an example of a vector space of functions of dimension five.

   (b) [25%] Let \( S \) be the vector space of all column vectors
   \[
   \begin{pmatrix}
   x_1 \\
   x_2 \\
   x_3
   \end{pmatrix}
   \]
   and let \( V \) be the subset of \( S \) given by
   the equation \( 2x_2 = 3(x_1 - x_3) \). Prove that \( V \) is a subspace of \( S \).
   Edwards and Penney Theorem 2 may be referenced in the proof, in order to shorten details.
   If you cite Theorem 2, then please state the Theorem.

   (c) [50%] Find a basis for the subspace of \( \mathbb{R}^3 \) given by the system of equations
   \[
   \begin{align*}
   x + 4y - 2z &= 0, \\
   x + 2y - 3z &= 0, \\
   2y + z &= 0,
   \end{align*}
   \]

   \( \mathbf{a} \) \( V = \) all linear combinations of atoms \( 1, x, x^2, x^3, x^4 \)

   \( \mathbf{b} \) Apply Thm 2, EGP. Define \( A = \begin{pmatrix} 3 & -2 & -3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \).
   Then \( A \mathbf{x} = \mathbf{0} \) defines \( V \). By Thm 2, \( V \) is a subspace.

   \( \mathbf{c} \) \[
   \begin{pmatrix}
   1 & 4 & -2 \\
   1 & 2 & -3 \\
   0 & 2 & 1
   \end{pmatrix}
   \begin{pmatrix}
   0 \\
   0 \\
   0
   \end{pmatrix}
   \]
   \[
   \begin{pmatrix}
   1 & 1 & -2 \\
   0 & -2 & -1 \\
   0 & 2 & 1
   \end{pmatrix}
   \begin{pmatrix}
   0 \\
   0 \\
   0
   \end{pmatrix}
   \]
   \[
   \begin{pmatrix}
   1 & 1 & -2 \\
   0 & 2 & 1 \\
   0 & 0 & 0
   \end{pmatrix}
   \begin{pmatrix}
   0 \\
   0 \\
   0
   \end{pmatrix}
   \]
   \[
   \begin{pmatrix}
   1 & 0 & -4 \\
   0 & 2 & 1 \\
   0 & 0 & 0
   \end{pmatrix}
   \begin{pmatrix}
   0 \\
   0 \\
   0
   \end{pmatrix}
   \]
   General solution:
   \[
   \begin{cases}
   x = 4t_1 \\
   y = -t_1/2 \\
   z = t_1
   \end{cases}
   \]
   Basis = \( \{ t_1, (\text{Gen. Sol.}) \} \)

Use this page to start your solution. Attach extra pages as needed, then staple.
3. (Independence) Do only two of the following.

(a) [50%] Let \( \mathbf{u} = \begin{pmatrix} 1 \\ -1 \\ 1 \\ 0 \end{pmatrix} \), \( \mathbf{v} = \begin{pmatrix} 2 \\ 1 \\ 0 \\ -1 \end{pmatrix} \), \( \mathbf{w} = \begin{pmatrix} 1 \\ 2 \\ -1 \\ 0 \end{pmatrix} \). State and apply a test that decides independence or dependence of the list of vectors \( \mathbf{u}, \mathbf{v}, \mathbf{w} \).

(b) [50%] State the pivot theorem [10%], then extract from the list below a largest set of independent vectors [40%].

\[
\begin{align*}
\mathbf{a} &= \begin{pmatrix} 1 \\ 1 \\ 0 \\ 3 \end{pmatrix}, \\
\mathbf{b} &= \begin{pmatrix} 1 \\ -1 \\ 0 \\ -1 \end{pmatrix}, \\
\mathbf{c} &= \begin{pmatrix} 2 \\ -2 \\ 0 \\ -2 \end{pmatrix}, \\
\mathbf{d} &= \begin{pmatrix} 5 \\ -3 \\ 0 \\ -1 \end{pmatrix}, \\
\mathbf{e} &= \begin{pmatrix} 2 \\ 0 \\ 0 \\ 2 \end{pmatrix}, \\
\mathbf{f} &= \begin{pmatrix} 3 \\ -1 \\ 0 \\ 1 \end{pmatrix}.
\end{align*}
\]

(c) [50%] Assume that matrix \( D \) is invertible. Prove:

If \( D\mathbf{x}_1, D\mathbf{x}_2, \ldots, D\mathbf{x}_n \) are independent, then \( \mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n \) are independent.

\[\text{A} \quad \text{Vectors } \mathbf{u}, \mathbf{v}, \mathbf{w} \text{ are independent } \Leftrightarrow \text{ rank}(\text{aug}(\mathbf{u}, \mathbf{v}, \mathbf{w})) = 3.\]

\[
\begin{pmatrix}
-1 & 2 & 1 \\
1 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix} \|_2 = \begin{pmatrix}
1 & 2 & 1 \\
0 & -2 & -2 \\
0 & 0 & 0
\end{pmatrix} \|_2 = \begin{pmatrix}
1 & 2 & 1 \\
0 & 1 & 0 \\
0 & 0 & 0
\end{pmatrix} \text{ Rank} = 2 \Rightarrow \boxed{\text{dependent}}
\]

\[\text{B} \quad \text{Pivot Theorem: The pivot columns of } A \text{ are independent and any other column of } A \text{ is dependent on them.}\]

\[
\begin{pmatrix}
1 & -1 & 2 & 5 & 2 & 3 \\
0 & 0 & -3 & 0 & 0 & 1 \\
3 & -1 & -2 & 0 & 2 & 1
\end{pmatrix} \|_2 = \begin{pmatrix}
1 & 1 & 2 & 5 & 2 & 3 \\
0 & -2 & -4 & -8 & -2 & -4 \\
0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix} \|_2 = \begin{pmatrix}
1 & 1 & 2 & 5 & 2 & 3 \\
0 & 2 & 4 & 8 & 2 & 4 \\
0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
\]

pivot cols are 1, 2. \boxed{\text{Independent cols of } A \text{ are 1, 2}}

\[\text{C} \quad \text{Let } \sum_{i=1}^n c_i \mathbf{x}_i = \mathbf{0}. \text{ Then} \]

\[
D \left( \sum_{i=1}^n c_i \mathbf{x}_i \right) = \mathbf{0}
\]

\[\Rightarrow \sum_{i=1}^n c_i D \mathbf{x}_i = \mathbf{0}
\]

\[\Rightarrow c_1, \ldots, c_n \text{ are zero, by independence of } D\mathbf{x}_1, \ldots, D\mathbf{x}_n.
\]

Use this page to start your solution. Attach extra pages as needed, then staple.
4. (determinants and elementary matrices)
   (a) [50%] Assume given invertible $3 \times 3$ matrices $A$, $B$. Suppose $B^2 = E_3E_2E_1A^2$ and $E_1$, $E_2$, $E_3$ are
elementary matrices representing respectively a swap, a combination and a multiply by 2. Compute the
possible values of $\det(-AB^{-1})$.
   (b) [50%] Let $A$, $B$ and $C$ be three $5 \times 5$ matrices such that $ABC$ contains two rows all of whose entries
are sevens. Explain precisely why at least one of the three matrices has zero determinant.

(a) The 2 is a typo. Assume multiply by $\times$ from $E_3$ instead of 2.

$$
\det (-A B^{-1}) = \det ((-I)(A)(B^{-1})) = \det (-I) \det (A B^{-1}) = \det (A B^{-1})
$$

prod rule for determinants

$$
\det B^2 = \det (E_3 E_2 E_1 A^2)
$$

$$
(\det B)^2 = \det (E_3) \det (E_2) \det (E_1) (\det (A))^2
$$

prod rule

$$
(\det B)^2 = (x)(1)(-1) (\det (A))^2
$$

$$
\det (A B^{-1}) = \det (A) \det (B^{-1}) = \det (A) \frac{1}{\det (B)} = \frac{\det A}{\det B}
$$

because $\det B B^{-1} = \det I$

$$
= \pm \sqrt{\left(\frac{\det A}{\det B}\right)^2}
$$

$$
= \pm \sqrt{\frac{1}{-x}}
$$

any correct sequence of steps was given full credit.
A retest was scheduled for those who were stopped by
The typo ($x=-2$ was the fix, but never applied).
Instructions: This in-class exam is 10 minutes. No calculators, notes, tables or books. The score on this problem replaces any previous score.

1. (Determinants and elementary matrices)
   (a) [50%] Assume given an invertible 4 × 4 matrix A. Suppose \( \text{rref}(A) = E_4E_3E_2E_1A \) and \( E_1, E_2, E_3, E_4 \) are elementary matrices representing respectively a swap, a combination, a swap and a multiply by -3. Compute \( \det(-2A^{-2}) \).
   (b) [50%] Let \( A \) be a three 5 × 5 matrix which contains one row all of whose entries are \( π \) and another row all of whose entries are \( e^{π} \). Explain precisely why \( Ax = 0 \) has infinitely many solutions \( x \).

\[ A \text{ invertible 4×4 } \iff \text{rref}(A) = I. \text{ Then} \]

\[ I = E_4E_3E_2E_1A \]

\[ \det(I) = \det(E_4E_3E_2E_1A) \]

\[ 1 = \det(E_4) \det(E_3) \det(E_2) \det(E_1) \det(A) \]

\[ 1 = \left(-3\right)(-1)(1)(-1) \det(A) \]

\[ \det(-2A^{-2}) = \det\left((-2I)(A)^{-2}\right) \]

\[ = \det(-2I) \det(A^{-1}) \det(A^{-1}) \]

\[ = (-2)^4 \left(\det(A)\right)^{-1} \left(\det(A)\right)^{-1} \]

\[ = \frac{16}{\left(\det(A)\right)^2} \]

\[ = \frac{16}{\left(-\frac{1}{3}\right)^2} \]

\[ = \boxed{144} \]

(b) The matrix has proportional rows, so a combo will produce a row of zeros. By the four rules for determinants, combo, plus "Thm a zero row \( \Rightarrow \) zero determinant", it follows that \( \det(A) = 0 \).

By Cramer's rule: There are infinitely many solutions to \( Ax = 0 \).
5. (inverses and Cramer's rule)

(a) [50%] Determine all values of $x$ and $y$ for which $A^{-1}$ fails to exist: $A = \begin{pmatrix} 1 & x-1 & 0 \\ 2 & 0 & -3 \\ 0 & 2y & 1 \end{pmatrix}$.

(b) [50%] Solve for $z$ in $Au = b$ by Cramer's rule: $A = \begin{pmatrix} 1 & 2 & 0 \\ 3 & 0 & 4 \\ 5 & 6 & 7 \end{pmatrix}$, $u = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$, $b = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$.

\[ A^{-1} \text{ fails to exist } \iff \det(A) = 0 \]
\[ \iff \begin{vmatrix} 1 & x-1 & 0 \\ 2 & 0 & -3 \\ 0 & 2y & 1 \end{vmatrix} = 0 \]
\[ \iff (1)(6y) - (x-1)(2-0) = 0 \]
\[ \iff 6y - 2x + 2 = 0 \]

\[ z = \frac{\Delta_3}{\Delta} \]
\[ z = \frac{2y}{-26} \]

\[ \Delta = \begin{vmatrix} 1 & 2 & 0 \\ 3 & 0 & 4 \\ 5 & 6 & 7 \end{vmatrix} \]
\[ = (1)(0 - 24) - 2(21 - 20) \]
\[ = -24 - 2 \]
\[ = -26 \]

\[ \Delta_3 = \begin{vmatrix} 1 & 2 & 1 \\ 3 & 0 & 0 \\ 5 & 6 & -1 \end{vmatrix} \]
\[ = (1)(0 - 0) - 2(-3 - 0) + 1(18) \]
\[ = 6 + 18 \]
\[ = 24 \]