Name. KEY

1. (rref)

Determine a, b such that the system has (1) infinitely many solutions, (2) no solutions.

$$\begin{array}{rcrcrcl} x & + & 6y & + & z & = & 1+a \\ 5x & + & 3y & + & 2z & = & 3+3a \\ 6x & + & 9y & + & 3bz & = & 2+a \end{array}$$

$$\left(\begin{array}{ccc|c} 1 & 6 & 1 & 1+a \\ 5 & 3 & 2 & 3+3a \\ 6 & 9 & 3b & 2+a \end{array}\right)$$

$$\cong \left(\begin{array}{ccc|c} 1 & 6 & 1 & 1+a \\ 0 & -27 & -3 & -2-2a \\ 0 & -27 & 3b-6 & -4-5a \end{array}\right)$$

$$\cong \left(\begin{array}{ccc|c} 1 & 6 & 1 & 1+a \\ 0 & 27 & 3 & 2+2a \\ 0 & 0 & 3b-3 & -2-3a \end{array}\right)$$

answer (1) : ∞-many sols $\iff$ last row is all zeros

$\iff$ $b=1$, $a=-2/3$

answer (2) : No solutions $\iff$ signal equation

$\iff$ last row is $0\ 0\ 0\ x$ with $x \neq 0$

$\iff$ $b=1$, $a \neq -2/3$

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Name. KEY

2. (vector spaces)

(a) [25%] Give an example of a vector space of functions of dimension five.

(b) [25%] Let S be the vector space of all column vectors $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ and let V be the subset of S given by the equation $2x_2 = 3(x_1 - x_3)$. Prove that V is a subspace of S. Edwards and Penney Theorem 2 may be referenced in the proof, in order to shorten details. If you cite Theorem 2, then please state the Theorem.

(c) [50%] Find a basis for the subspace of $\mathcal{R}^3$ given by the system of equations

$$\begin{aligned} x + 4y - 2z &= 0, \\ x + 2y - 3z &= 0, \\ 2y + z &= 0, \end{aligned}$$

(a) $V =$ all linear combinations of atoms $1, x, x^2, x^3, x^4$

(b) Apply Thm 2, E&P. Define $A = \begin{pmatrix} 3 & -2 & -3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. Then $A\vec{x} = \vec{0}$ defines V. By Thm 2, V is a subspace.

(c) $\left(\begin{array}{ccc|c} 1 & 4 & -2 & 0 \\ 1 & 2 & -3 & 0 \\ 0 & 2 & 1 & 0 \end{array}\right)$

$\cong \left(\begin{array}{ccc|c} 1 & 4 & -2 & 0 \\ 0 & -2 & -1 & 0 \\ 0 & 2 & 1 & 0 \end{array}\right)$

$\cong \left(\begin{array}{ccc|c} 1 & 4 & -2 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$

$\cong \left(\begin{array}{ccc|c} 1 & 0 & -4 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$

Gen. sol.

$$\begin{cases} x = 4t_1 \\ y = -t_1/2 \\ z = t_1 \end{cases}$$

Basis $= \{ \partial_{t_1}(\text{Gen. sol.}) \}$

$= \boxed{\left\{ \begin{pmatrix} 4 \\ -1/2 \\ 1 \end{pmatrix} \right\}}$

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3. (independence) Do **only two** of the following.

(a) [50%] Let $\mathbf{u} = \begin{pmatrix} 1 \\ -1 \\ 1 \\ 0 \end{pmatrix}$, $\mathbf{v} = \begin{pmatrix} 2 \\ 1 \\ 0 \\ 0 \end{pmatrix}$, $\mathbf{w} = \begin{pmatrix} 1 \\ 2 \\ -1 \\ 0 \end{pmatrix}$. State and apply a test that decides independence or dependence of the list of vectors $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$.

(b) [50%] State the pivot theorem [10%], then extract from the list below a largest set of independent vectors [40%].

$$\mathbf{a} = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 3 \end{pmatrix}, \mathbf{b} = \begin{pmatrix} 1 \\ -1 \\ 0 \\ -1 \end{pmatrix}, \mathbf{c} = \begin{pmatrix} 2 \\ -2 \\ 0 \\ -2 \end{pmatrix}, \mathbf{d} = \begin{pmatrix} 5 \\ -3 \\ 0 \\ -1 \end{pmatrix}, \mathbf{e} = \begin{pmatrix} 2 \\ 0 \\ 0 \\ 2 \end{pmatrix}, \mathbf{f} = \begin{pmatrix} 3 \\ -1 \\ 0 \\ 1 \end{pmatrix}.$$

(c) [50%] Assume that matrix D is invertible. Prove:

If $D\mathbf{x}_1$, $D\mathbf{x}_2$, ..., $D\mathbf{x}_n$ are independent, then $\mathbf{x}_1$, $\mathbf{x}_2$, ..., $\mathbf{x}_n$ are independent.

(a) Vectors $\vec{u}, \vec{v}, \vec{w}$ are independent $\iff$ rank$(\text{aug}(\vec{u}, \vec{v}, \vec{w})) = 3$.

$$\begin{pmatrix} 1 & 2 & 1 \\ -1 & 1 & 2 \\ 1 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix} \cong \begin{pmatrix} 1 & 2 & 1 \\ 0 & 3 & 3 \\ 0 & -2 & -2 \\ 0 & 0 & 0 \end{pmatrix} \cong \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

rank$=2 \Rightarrow \boxed{\text{dependent}}$

(b) Pivot Theorem: The pivot columns of A are independent and any other column of A is dependent on them.

$$\begin{pmatrix} 1 & 1 & 2 & 5 & 2 & 3 \\ 1 & -1 & -2 & -3 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 3 & -1 & -2 & -1 & 2 & 1 \end{pmatrix} \cong \begin{pmatrix} 1 & 1 & 2 & 5 & 2 & 3 \\ 0 & -2 & -4 & -8 & -2 & -4 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -4 & -8 & -16 & -4 & -8 \end{pmatrix} \cong \begin{pmatrix} 1 & 1 & 2 & 5 & 2 & 3 \\ 0 & 2 & 4 & 8 & 2 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

pivot cols are 1, 2. $\boxed{\text{Independent cols of } A \text{ are } 1, 2}$

(c) Let $\sum_{i=1}^{n} c_i \vec{x}_i = \vec{0}$. Then

$$D\left(\sum_1^n c_i \vec{x}_i\right) = \vec{0}$$

$$\Rightarrow \sum_1^n c_i D\vec{x}_i = \vec{0}$$

$\Rightarrow$ $c_1, \ldots, c_n$ are zero, by independence of $D\vec{x}_1, \ldots, D\vec{x}_n$.

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Name. KEY

4. **(determinants and elementary matrices)**

(a) [50%] Assume given invertible 3×3 matrices A, B. Suppose $B^2 = E_3E_2E_1A^2$ and E_1, E_2, E_3 are elementary matrices representing repectively a swap, a combination and a multiply by 2. Compute the possible values of $\det(-AB^{-1})$.

(b) [50%] Let A, B and C be three 5×5 matrices such that ABC contains two rows all of whose entries are sevens. Explain precisely why at least one of the three matrices has zero determinant.

(a) The 2 is a typo. Assume multiply by x for E_3 instead of 2.

$$\det(-AB^{-1}) = \det((-I)(A)(B^{-1}))$$
$$= \det(-I)\det(AB^{-1}) \quad \text{prod rule for determinants}$$
$$= (-1)^3 \det(AB^{-1}) \quad \text{triangular rule, } 3\times 3$$

$$\det B^2 = \det(E_3E_2E_1A^2)$$
$$(\det B)^2 = \det(E_3)\det(E_2)\det(E_1)(\det(A))^2 \quad \text{prod rule}$$
$$(\det B)^2 = (x)(1)(-1)(\det(A))^2$$

$$\det(AB^{-1}) = \det(A)\det(B^{-1})$$
$$= \det(A)\frac{1}{\det(B)} \quad \text{because } \det BB^{-1} = \det I$$
$$= \frac{\det A}{\det B}$$
$$= \pm\sqrt{\left(\frac{\det A}{\det B}\right)^2}$$
$$= \pm\sqrt{\frac{1}{-x}}$$

any correct sequence of steps was given full credit. A retest was scheduled for those who were stopped by the typo ($x=-2$ was the fix, but never applied).

Use this page to start your solution. Attach extra pages as needed, then staple.

Applied Differential Equations 2250

Midterm Exam 2, Problem 4 Re-Test, 10:45am

Exam date: Tuesday, 4 April 2006

Instructions: This in-class exam is 10 minutes. No calculators, notes, tables or books. The score on this problem replaces any previous score.

4. **(determinants and elementary matrices)**
(a) [50%] Assume given an invertible 4×4 matrix A. Suppose $\mathbf{rref}(A) = E_4E_3E_2E_1A$ and E_1, E_2, E_3, E_4 are elementary matrices representing repectively a swap, a combination, a swap and a multiply by -3. Compute $\det(-2A^{-2})$.
(b) [50%] Let A be a three 5×5 matrix which contains one row all of whose entries are π and another row all of whose entries are e^{π}. Explain precisely why $A\mathbf{x} = \mathbf{0}$ has infinitely many solutions $\mathbf{x}$.

(a) A invertible $4\times4 \iff \mathrm{rref}(A) = I$. Then

$$I = E_4 E_3 E_2 E_1 A$$

$$\det(I) = \det(E_4 E_3 E_2 E_1 A)$$

$$1 = \det(E_4)\det(E_3)\det(E_2)\det(E_1)\det(A)$$ prod. Thm. for determinants.

$$1 = (-3)(-1)(1)(-1)\det(A)$$

$$\det(-2A^{-2}) = \det((-2I)(A)^{-2})$$

$$= \det(-2I)\det(A^{-1})\det(A^{-1})$$ prod. Thm

$$= (-2)^4(\det(A))^{-1}(\det(A))^{-1}$$ Triang rule, 4x4

$$= \frac{16}{(\det(A))^2}$$

$$= \frac{16}{\left(\frac{1}{-3}\right)^2}$$

$$= \boxed{144}$$

(b) The matrix has proportional rows, so a combo will produce a row of zeros. By the four rules for determinants, combo, plus "Thm a zero row $\Rightarrow$ zero determinant" it follows that $\det(A)=0$. By Cramer's rule there are ∞-many solutions to $A\vec{x}=\vec{0}$.

Use this page to start your solution. Attach extra pages as needed, then staple.

Name. KEY

5. (inverses and Cramer's rule)

(a) [50%] Determine all values of x and y for which A^{-1} fails to exist: $A = \begin{pmatrix} 1 & x-1 & 0 \\ 2 & 0 & -3 \\ 0 & 2y & 1 \end{pmatrix}$.

(b) [50%] Solve for z in $A\mathbf{u} = \mathbf{b}$ by Cramer's rule: $A = \begin{pmatrix} 1 & 2 & 0 \\ 3 & 0 & 4 \\ 5 & 6 & 7 \end{pmatrix}$, $\mathbf{u} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$.

(a) A^{-1} fails to exist $\iff \det(A) = 0$

$$\iff \begin{vmatrix} 1 & x-1 & 0 \\ 2 & 0 & -3 \\ 0 & 2y & 1 \end{vmatrix} = 0$$

$$\iff (1)(6y) - (x-1)(2-0) = 0$$

$$\iff \boxed{6y - 2x + 2 = 0}$$

(b) $z = \dfrac{\Delta_3}{\Delta}$

$$\boxed{z = \frac{24}{-26}}$$

$$\Delta = \begin{vmatrix} 1 & 2 & 0 \\ 3 & 0 & 4 \\ 5 & 6 & 7 \end{vmatrix}$$

$= (1)(0-24) - 2(21-20)$

$= -24 - 2$

$= -26$

$$\Delta_3 = \begin{vmatrix} 1 & 2 & 1 \\ 3 & 0 & 0 \\ 5 & 6 & -1 \end{vmatrix}$$

$= (1)(0-0) - 2(-3-0) + 1(18)$

$= 6 + 18$

$= 24$

Use this page to start your solution. Attach extra pages as needed, then staple.