Differential Equations and Linear Algebra 2250 [7:30]
Midterm Exam 1
Tuesday, 14 February 2006

Instructions: This in-class exam is 50 minutes. No calculators, notes, tables or books. No answer check is expected. Details count 75%. The answer counts 25%.

1. Quadrature Equation

Solve for the general solution $y(x)$ in the equation $y' = (1-x)e^{-2x} + 2 \tan x + \frac{32x^3}{1+4x^2}$.

\[
\int y' \, dx = \int F(x) \, dx \\
\frac{dy}{dx} = C + I_1 + I_2 + I_3 \\
I_1 = \int (1-x)e^{-2x} \, dx \\
= (1-x) \frac{e^{-2x}}{-2} - \int \frac{e^{-2x}}{2} (-d(1-x)) \\
= \left[ \frac{1}{4} e^{-2x} + \frac{1}{2} e^{-2x} \right] \\
I_2 = \int 2 \tan x \, dx \\
= \int \frac{2 \sin x \, dx}{\cos x} \\
= -2 \ln |\cos x| \\
I_3 = \int \frac{32x^3}{1+4x^2} \, dx \\
= \int 8x \, dx - \int \frac{8x \, dx}{1+4x^2} \\
= \left[ 4x^2 - \ln(1+4x^2) \right]
\]

9 persons got 100
2. (Separable Equation Test)
The problem \( y' = f(x, y) \) is said to be separable provided \( f(x, y) = F(x)G(y) \) for some functions \( F \) and \( G \).

(a) [75%] Check (X) the problems that can be put into separable form, but don’t supply any details.

\[
\begin{array}{|c|c|}
\hline
\text{X} & y' = -y(xy+1)+(x+1)y^2 & \text{X} & yy' = xy^2 + x \\
\hline
\text{X} & y' = x + e^y & \text{X} & y' + y = 10 \\
\hline
\end{array}
\]

(b) [25%] Give a test which can verify that an equation is not separable. Use the test to show that \( y' = x + y^2 \) is not separable.

\[
\begin{align*}
\left[ 10 \right] \text{ Test: } & \text{ Define } F(x) = \frac{f(x, y_0)}{f(x_0, y_0)}, \ G(y) = f(x_0, y) \text{ in } f(x_0, y_0) \neq 0. \\
& \text{ If } F \circ G = f, \text{ then } y' = f(x, y) \text{ is separable, and} \\
& \text{ Conversely.} \\
\end{align*}
\]

\[\text{Application:}\]

\[ F(x) = \frac{f(x, 1)}{f(0, 1)} = x + 1 \]

\[ G(y) = f(0, y) = y^2 \]

\[ F \circ G = (x + 1)y^2 \]

\[ = xy^2 + y^2 \]

\[ \neq x + y^2 \]

\[ \therefore F \circ G \neq f \text{ and the equation } y' = x + y^2 \text{ is not separable.} \]

Use this page to start your solution. Attach extra pages as needed, then staple.
3. (Solve a Separable Equation)
Given \( y^2y' = \frac{2x^2 + 4x}{1 + x^2} (125 - 8y^3) \).

(a) Find all equilibrium solutions.
(b) Find the non-equilibrium solution in implicit form.
To save time, do not solve for \( y \) explicitly.

(a) \( G(y) = \frac{125 - 8y^3}{y^2} \)
\[ G(c) = 0 \Leftrightarrow c = \frac{5}{2} \]
\[ y = \frac{5}{2} \]
is the only equilibrium sol.

(b) \[ \frac{y^2y'}{125 - 8y^3} = \frac{2x^2 + 4x}{1 + x^2} \]
\[ = 2 + \frac{4x-2}{1 + x^2} \]
\[ \int \frac{y^2y'}{125 - 8y^3} \, dx = \int 2 \, dx + 4 \int \frac{x \, dx}{1 + x^2} - \int \frac{2 \, dx}{1 + x^2} \]
\[ \frac{-1}{24} \ln |125 - 8y^3| = 2x + 2 \ln (1 + x^2) - 2 \text{arctan}(x) + C \]
4. (Linear Equations)
(a) [60%] Solve $2v'(t) = -32 + \frac{2}{2t+1} v(t)$, $v(0) = -16$. Show all integrating factor steps.
(b) [30%] Solve $\sqrt{x+1} y'(x) = y(x)$. The answer contains symbol $c$.
(c) [10%] The problem $\sqrt{x+1} y'(x) = y(x) - 2$ can be solved using the answer $y_h$ from (b) plus superposition $y = y_h + y_p$. Find $y_p$. Hint: If you cannot write the answer in 10 seconds, then return here after finishing all problems on the exam.

\[ v' - \frac{1}{2t+1} v = -16 \]
\[ v(0) = -16 \]

\[ Q = e^{\int -\frac{dt}{\sqrt{t+1}}} \]
\[ = e^{-\frac{1}{2} \ln |2t+1|} \]
\[ = (2t+1)^{-1/2} \quad \text{for} \ t \geq 0 \]

\[ (Qv)'/Q = -16 \quad \text{Replace in DE} \]

\[ Qv = -16 \int Q \]
\[ Qv = -16 \left( \frac{2t+1}{2} \right)^{1/2} \left( \frac{1}{2} \right) + C \]
\[ v = -16 \left( 2t+1 \right) + C(2t+1)^{1/2} \]
\[ v(0) = -16 \quad \Rightarrow \quad c = 0 \]
\[ v(t) = -32t - 16 \]

\[ y' - (x+1)^{1/2} y = 0 \]
\[ Q = e^{\int (x+1)^{1/2} dx} \]
\[ = e^{-2(x+1)^{1/2}} \]
\[ (Qy)'/Q = 0 \]
\[ y = c/Q \]
\[ y = ce^{2\sqrt{x+1}} \]

\[ y_p = 2 \]

Use this page to start your solution. Attach extra pages as needed, then staple.
5. (Stability)
   (a) [50%] Draw a phase line diagram for the differential equation
   \[
   \frac{dx}{dt} = (2 - \sqrt[3]{x})^3 (1 - 3x)(9x^2 - 1)^8.
   \]
   Expected in the diagram are equilibrium points and signs of \(x'\) (or flow direction markers < and >).

   (b) [50%] Draw a phase diagram using the phase line diagram of (a). Add these labels as appropriate: funnel, spout, node, source, sink, stable, unstable. Show at least 8 threaded curves. A direction field is not expected or required.

Equilibrium solutions are found from the equation

\[
(2 - x^{\frac{1}{3}})^3 (1 - 3x)(1 + 3x)^8 = 0
\]

\[
x = 8, \quad x = \frac{1}{3}, \quad x = -\frac{1}{3}
\]

Use this page to start your solution. Attach extra pages as needed, then staple.