Name. KEY

Differential Equations and Linear Algebra 2250 [7:30]

Midterm Exam 1
Tuesday, 14 February 2006

Instructions: This in-class exam is 50 minutes. No calculators, notes, tables or books. No answer check is expected. Details count 75%. The answer counts 25%.

1. (Quadrature Equation)

Solve for the general solution $y(x)$ in the equation $y' = (1-x)e^{-2x} + 2\tan x + \dfrac{32x^3}{1+4x^2}$.

$$\int y'\,dx = \int F(x)\,dx$$
$$y = C + I_1 + I_2 + I_3$$
$$I_1 = \int (1-x)e^{-2x}dx$$

parts
$u = 1-x, \quad dv = e^{-2x}dx$

$$= (1-x)\frac{e^{-2x}}{-2} - \int \frac{e^{-2x}}{-2}(-dx)$$
$$= \boxed{-\frac{1}{4}e^{-2x} + \frac{x}{2}e^{-2x}}$$

$$I_2 = \int 2\tan x\,dx$$
$$= \int \frac{2\sin x\,dx}{\cos x}$$
$$= \boxed{-2\ln|\cos x|}$$

$$I_3 = \int \frac{32x^3}{1+4x^2}dx$$
$$= \int 8x\,dx - \int \frac{8x}{1+4x^2}dx$$
$$= \boxed{4x^2 - \ln(1+4x^2)}$$

9 persons got 100

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2. (Separable Equation Test)
The problem $y' = f(x,y)$ is said to be separable provided $f(x,y) = F(x)G(y)$ for some functions F and G.

(a) [75%] Check ($\boxed{\text{X}}$) the problems that can be put into separable form, but don't supply any details.

X	$y' = -y(xy+1)+(x+1)y^2$	X	$yy' = xy^2 + x$
	$y' = x + e^y$	X	$y' + y = 10$

[8 each]

(b) [25%] Give a test which can verify that an equation is not separable. Use the test to show that $y' = x + y^2$ is not separable.

[10] Test: Define $F(x) = \dfrac{f(x,y_0)}{f(x_0,y_0)}$, $G(y) = f(x_0,y)$ for $f(x_0,y_0) \neq 0$.

If $FG = f$, then $y' = f(x,y)$ is separable, and conversely.

Application:

$$F(x) = \frac{f(x,1)}{f(0,1)} = x+1 \qquad G(y) = f(0,y) = y^2$$

$$FG = (x+1)y^2 = xy^2 + y^2 \neq x + y^2$$

$\therefore$ $FG \neq f$ and the equation $y' = x + y^2$ is not separable

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3. (Solve a Separable Equation)

Given $y^2y' = \dfrac{2x^2+4x}{1+x^2}(125-8y^3)$.

(a) Find all equilibrium solutions.

(b) Find the non-equilibrium solution in implicit form.

To save time, **do not solve** for y explicitly.

(a) $G(y) = \dfrac{125-8y^3}{y^2}$

$G(c)=0 \iff c = \dfrac{5}{2}$

$\boxed{y = \frac{5}{2}}$ is the only equilibrium sol.

(b) $$\frac{y^2y'}{125-8y^3} = \frac{2x^2+4x}{1+x^2}$$

$$= 2 + \frac{4x-2}{1+x^2}$$

$$\int \frac{y^2y'\,dx}{125-8y^3} = \int 2\,dx + 4\int \frac{x\,dx}{1+x^2} - \int \frac{2\,dx}{1+x^2}$$

$$\boxed{-\frac{1}{24}\ln|125-8y^3| = 2x + 2\ln(1+x^2) - 2\arctan(x) + C}$$

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4. (Linear Equations)

(a) [60%] Solve $2v'(t) = -32 + \frac{2}{2t+1}v(t)$, $v(0) = -16$. Show all integrating factor steps.

(b) [30%] Solve $\sqrt{x+1}\,y'(x) = y(x)$. The answer contains symbol c.

(c) [10%] The problem $\sqrt{x+1}\,y'(x) = y(x) - 2$ can be solved using the answer y_h from (b) plus superposition $y = y_h + y_p$. Find y_p. Hint: If you cannot write the answer in 10 seconds, then return here after finishing all problems on the exam.

(a) $v' - \frac{1}{2t+1}v = -16$, $v(0) = -16$

$Q = e^{\int -dt/(2t+1)}$

$= e^{-\frac{1}{2}\ln|2t+1|}$

$= (2t+1)^{-1/2}$ for $t \geq 0$

$(Qv)'/Q = -16$ Replace in DE

$Qv = -16\int Q$ Quadrature Step

$Qv = -16\,(2t+1)^{1/2}/(1/2)\cdot\left(\frac{1}{2}\right) + c$

$v = -16\,(2t+1) + c\,(2t+1)^{1/2}$

$v(0) = -16 \Rightarrow c = 0$

$\boxed{v(t) = -32t - 16}$

(b) $y' - (x+1)^{-1/2}y = 0$

$Q = e^{-\int (x+1)^{-1/2}dx}$

$= e^{-2(x+1)^{1/2}}$

$(Qy)'/Q = 0$

$y = c/Q$

$\boxed{y = c\,e^{2\sqrt{x+1}}}$

(c) an equilibrium solution is $\boxed{y_p = 2}$

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5. (Stability)

(a) [50%] Draw a phase line diagram for the differential equation

$$dx/dt = (2 - \sqrt[3]{x})^3 (1 - 3x)(9x^2 - 1)^8.$$

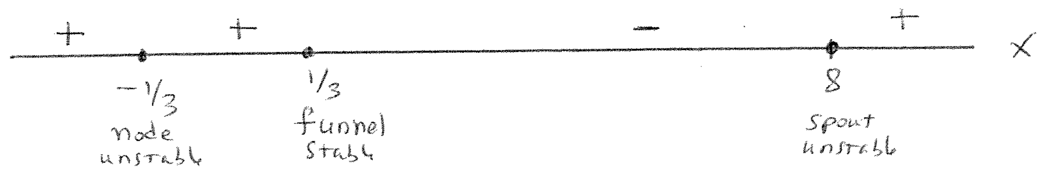
Expected in the diagram are equilibrium points and signs of x' (or flow direction markers $<$ and $>$).

(b) [50%] Draw a phase diagram using the phase line diagram of (a). Add these labels as appropriate: funnel, spout, node, source, sink, stable, unstable. Show at least 8 threaded curves. A direction field is not expected or required.

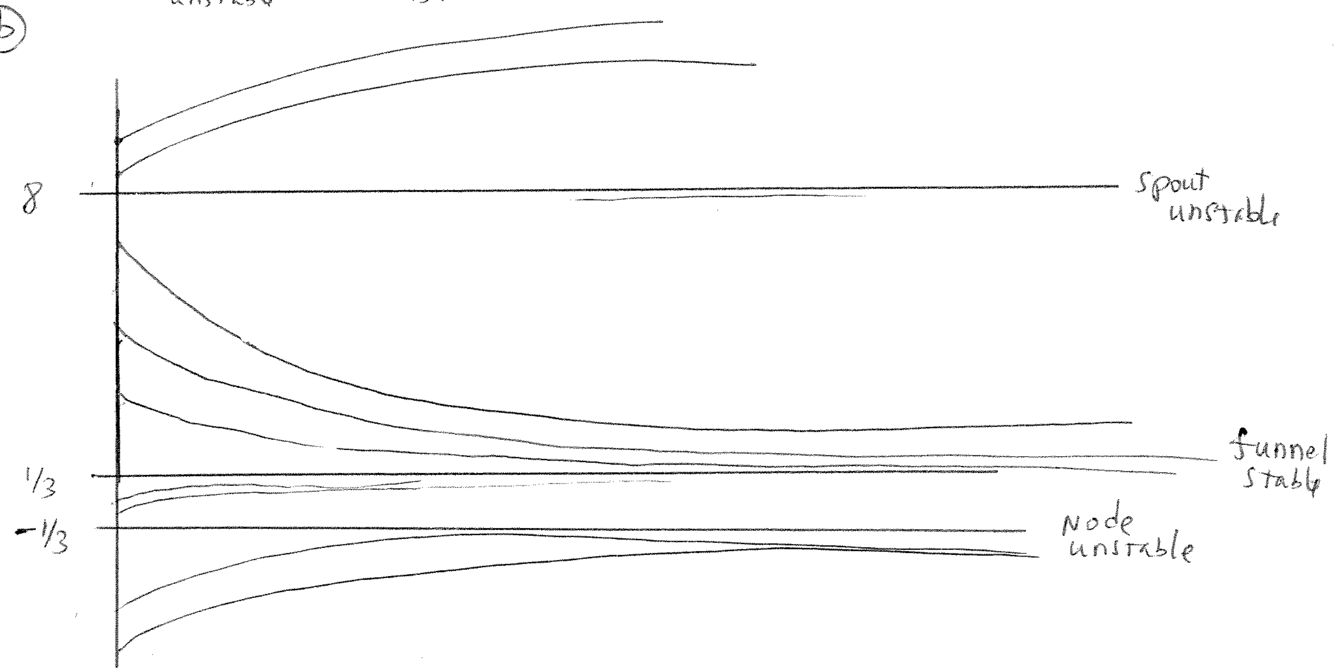
(a) Equilibrium solutions are found from the equation

$$(2 - x^{1/3})^3 (1-3x)^9 (1+3x)^8 = 0$$

$$x = 8, \quad x = 1/3, \quad x = -1/3$$



(b)



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