Differential Equations and Linear Algebra 2250 [10:45]
Midterm Exam 1
Version 3: Tuesday, 14 February 2006

Instructions: This in-class exam is 50 minutes. No calculators, notes, tables or books. No answer check is expected. Details count 75%. The answer counts 25%.

1. (Quadrature Equation)

Solve for the general solution $y(x)$ in the equation $y' = 2\cot x + \frac{1250x^3}{1 + 25x^2} + x\ln(1 + x^2)$.

\[ y = \int y' \, dx = \int F(x) \, dx \]
\[ y = 2 \int \cot x \, dx + \int \frac{1250x^3}{1 + 25x^2} \, dx + \int x\ln(1 + x^2) \, dx \]
\[ y = I_1 + I_2 + I_3 + C \quad \text{(work split below)} \]

\[ I_1 = 2 \int \cot x \, dx \]
\[ = 2 \int \frac{\cos x}{\sin x} \, du \]
\[ = 2 \ln |\sin x| \]

\[ I_2 = \int \frac{1250x^3 \, dx}{1 + 25x^2} \]
\[ = \int \left( 50x - \frac{50x}{1 + 25x^2} \right) \, dx \]
\[ = 25x^2 - \ln(1 + 25x^2) \]

\[ I_3 = \int x\ln(1 + x^2) \, dx \]
\[ = \int \ln(u) \, \frac{du}{2} \quad \text{with} \quad u = 1 + x^2, \: du = 2x \, dx \]
\[ = \frac{1}{2} \left( u \ln u - u \right) \]
\[ = \frac{1}{2} \left( (1 + x^2) \ln(1 + x^2) - (1 + x^2) \right) \]

\[ y = I_1 + I_2 + I_3 + C \]
\[ y = (2 \ln |\sin x|) + (25x^2 - \ln(1 + 25x^2)) + \left( \frac{1}{2} (1 + x^2) \ln(1 + x^2) - \frac{x}{2} \right) + C \]

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2. (Separable Equation Test)

The problem \( y' = f(x, y) \) is said to be separable provided \( f(x, y) = F(x)G(y) \) for some functions \( F \) and \( G \).

(a) [75%] Check the problems that can be put into separable form, but don't supply any details.

| \( x' = -y(2xy + 1) + (2x + 3)y^2 \) | \( xy' = xy^2 + 5x \) |
| \( y' = e^x + e^y \) | \( 3y' + 5y = 10 \) |

(b) [25%] State a test which can verify that an equation is not separable. Use the test to verify that \( y' = x + \sqrt{|y|} \) is not separable.

\[
\begin{array}{c|c}
\text{\( y' = -y(2xy + 1) + (2x + 3)y^2 \)} & \text{\( xy' = xy^2 + 5x \)} \\
\text{\( = -2xy^2 - y + 2xy + 3y^2 \)} & \text{\( = x(y^2 + 5) \)} \\
\text{\( = -y + 3y^2 \)} & \text{\( \text{sep.} \)} \\
\text{autonomous \( \Rightarrow \) \( \text{sep.} \)} & \\
\text{\( y' = e^x + e^y \)} & \text{\( 3y' + 5y = 10 \)} \\
\text{not \( \text{sep.} \)} & \text{\( y' = \frac{10 - 5y}{3} \)} \\
& \text{autonomous \( \Rightarrow \) \( \text{sep} \)}
\end{array}
\]

\( b \) Let \( F(x) = \frac{f(x, y_0)}{f(x_0, y_0)} \), \( G(y) = \frac{f(x_0, y)}{f(x_0, y_0)} \), \( f(x_0, y_0) \neq 0 \). Then \( FG + f \) implies \( y' = f(x, y) \) is not separable.

Application: Choose \( x_0 = 1, y_0 = 0 \). Then \( f(x, y) = x + \sqrt{|y|} \) implies \( F(x) = \frac{f(x, 0)}{f(1, 0)} = x \), \( G(y) = 1 + \sqrt{|y|} \). Then

\[
\begin{align*}
F &= x \left( 1 + \sqrt{|y|} \right) \\
&= x + x\sqrt{|y|} \\
&= x + \sqrt{y} = f
\end{align*}
\]

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3. (Solve a Separable Equation)

Given \( y^2 y' = \frac{2x^2 + 3x}{1 + x^2} \left( \frac{125}{64} - y^3 \right) \).

(a) Find all equilibrium solutions.

(b) Find the non-equilibrium solution in implicit form.

To save time, **do not solve** for \( y \) explicitly.

\[
\begin{align*}
\text{(a)} \quad F(x) &= \frac{2x^2 + 3x}{1 + x^2} \\
&= 2 + \frac{3x - 2}{1 + x^2} \\
G(y) &= \left( \frac{125}{64} - y^3 \right)^{\frac{1}{3}} \\
G'(y) &= 0 \implies y = \sqrt[3]{\frac{125}{64}} \quad \text{or} \quad y = \frac{5}{4} \\
\hline
\text{(b)} \quad \frac{y'}{G(y)} &= F(x) \\
\int \frac{y'^2}{G(y)} \, dx &= \int F(x) \, dx \\
\int \frac{y^2 \, y'}{\frac{125}{64} - y^3} \, dx &= \int \left( 2 + \frac{3}{2} \left( \frac{2x}{1 + x^2} \right) + \frac{2}{1 + x^2} \right) \, dx \\
-\frac{1}{3} \ln \left| \frac{125}{64} - y^3 \right| &= 2x + \frac{3}{2} \ln (1 + x^2) + 2 \arctan(x) + C
\end{align*}
\]

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4. (Linear Equations)

(a) [60%] Solve $2v'(t) = -32 + \frac{2}{3t+1} v(t)$, $v(0) = -8$. Show all integrating factor steps.

(b) [30%] Solve $2\sqrt{x} + 2 \frac{dy}{dx} = y$. The answer contains symbol $c$.

(c) [10%] The problem $2\sqrt{x} + 2y' = y - 5$ can be solved using the answer $y_h$ from (b) plus superposition $y = y_h + y_p$. Find $y_p$. Hint: If you cannot write the answer in a few seconds, then return here after finishing all problems on the exam.

\[ y' - \left( \frac{1}{3t+1} \right) v = -16 \quad \text{valid} \quad v(0) = -8 \]

\[ Q = e^{-\int \frac{dt}{3t+1}} \]

\[ = e^{-\frac{1}{3} \ln |3t+1|} \]

\[ = (3t+1)^{-1/3} \]

\[ (Qv)/Q = -16 \]

\[ c_n V = -16 \int (3t+1)^{-1/3} dt + c \]

\[ = -16 \int (3t+1)^{-1/3} \cdot \frac{1}{2} + c \]

\[ = -8(3t+1)^{2/3} + c \]

\[ v = -8(3t+1) + C (3t+1)^{1/3} \]

\[ v = -8 + C \]

\[ C = 0 \]

\[ v = -8(3t+1) \]

\[ v = -24t - 8 \]

(b) \[ y' - \frac{1}{2\sqrt{x+2}} y = 0 \]

\[ Q = e^{-\int \frac{dx}{\sqrt{x+2}}} \]

\[ = e^{-\sqrt{x+2}} \]

\[ (Qy)' = 0 \]

\[ y = \frac{c}{Q} \]

\[ y = ce^{-\sqrt{x+2}} \]

(c) \[ y_p = 5 \]

is an equilibrium solution.

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5. (Stability)
(a) [50%] Draw a phase line diagram for the differential equation

\[ \frac{dx}{dt} = 1000 \left( 2 - \sqrt[3]{x} \right)^3 \left( 2 + 3x \right) \left( 9x^2 - 4 \right)^8. \]

Expected in the diagram are equilibrium points and signs of \( x' \) (or flow direction markers < and >).
(b) [50%] Draw a phase diagram using the phase line diagram of (a). Add these labels as appropriate: funnel, spout, node, source, sink, stable, unstable. Show at least 8 threaded curves. A direction field is not expected or required.

Equilibrium solutions are found from

\[ 1000 \left( 2 - x^{\frac{1}{3}} \right)^3 \left( 2 + 3x \right) \left( 9x^2 - 4 \right)^8 = 0 \]

\[ x = 32, \quad x = -\frac{2}{3}, \quad x = \frac{2}{3} \]

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