

**Math 2250**  
**Maple Project 1**  
**January 2003**

**Due date:** See the internet due dates.

**References:** Edwards-Penney, pages 55–57. Code in `maple` appears in 2250mapleL1-S2003-text.mws at URL <http://www.math.utah.edu/~gustafso/>. This document: 2250mapleL1-S2003.pdf.

**The Problem.** The project is to determine an approximation for the indoor temperature  $u(t)$  in an unheated building. The model uses Newton cooling, insulation data and a formula for the ambient outside temperature  $A(t)$ .

**Assumptions and Notation.** Let the daily temperature in Salt Lake City vary from a low of  $m = 20$  degrees to a high of  $M = 50$  degrees with the low and high at 3am and 3pm, respectively. The building is assumed insulated, the average insulation factor being represented by a Newton cooling constant  $k$ , usually  $0.2 \leq k \leq 0.5$ . The ambient outside temperature is called  $A(t)$  and the inside temperature is called  $u(t)$ , where  $t$  is in hours measured from midnight  $t = 0$ . The number  $u_0$  is the temperature at midnight. Modeling assumes that no heat sources or heat sinks are present inside the building.

**Newton Cooling Model.** Newton's law of cooling is:

*The rate of change of the indoor temperature is proportional to the difference between the ambient and indoor temperatures.*

This implies that  $du/dt$  is proportional to  $A - u$ . Let  $A(t)$  be the ambient outside temperature and let  $k$  denote a constant. A model for the indoor temperature  $u(t)$  is given by  $du/dt = k(A - u)$ ,  $u(0) = u_0$ , which can be rearranged as

$$(1) \quad u'(t) + ku(t) = kA(t), \quad u(0) = u_0.$$

The number  $k$  is called the **insulation constant**.

**Ambient Temperature Model.** Let  $M = 50$ ,  $m = 20$ ,  $\omega_0 = \pi/12$ . The ambient temperature formula

$$A(t) = \frac{1}{2}(M + m) - \frac{1}{2}(M - m) \cos \omega_0(t - 3)$$

satisfies  $M = \max A(t) = A(15)$ ,  $m = \min A(t) = A(3)$  and  $A(t)$  is  $t$ -periodic of period 24 hours.

For use in `maple`, a function of two variables

$$AA(t, \omega) = \frac{1}{2}(M + m) - \frac{1}{2}(M - m) \cos \omega(t - 3)$$

is used to maintain the variable name  $\omega$  in displays.

**Indoor Temperature  $u(t)$ .** The integrating factor method for linear equations applies to find the general solution by these steps:

$$u' + ku = kA(t) \quad \text{Copy the differential equation. The integrating factor is } e^{kt}.$$

$$\frac{(e^{kt}u)'}{e^{kt}} = kA(t) \quad \text{Replace the left side.}$$

$$(e^{kt}u)' = kA(t)e^{kt} \quad \text{Clear fractions.}$$

$$e^{kt}u = u_0 + \int_0^t kA(x)e^{kx} dx \quad \text{Apply the method of quadrature and use } u(0) = u_0.$$

$$u = u_0e^{-kt} + e^{-kt} \int_0^t kA(x)e^{kx} dx \quad \text{Divide to isolate } u.$$

$$u = u_0e^{-kt} + k \int_0^t e^{k(x-t)} A(x) dx \quad \text{Exponential rule: } e^a e^b = e^{a+b}.$$

Let  $u_h(t) = u_0e^{-kt}$ , a solution of the homogeneous differential equation  $u' + ku = 0$ . Let  $u_p(t) = k \int_0^t e^{k(x-t)} A(x) dx$ , a particular solution of the nonhomogeneous differential equation  $u' + ku = kA(t)$ . Then the indoor temperature  $u = u_h + u_p$  depends on the time  $t$ , the initial temperature  $u_0$ , the insulation constant  $k$  and the frequency  $\omega$  (which is fixed at  $\pi/12$ ). Write  $u = u(t, u_0, k, \omega)$  to emphasize the dependence. In `maple`, advantages exist for adding the variable name  $\omega$ , which is later set to value  $\omega_0 = \pi/12$ . Write  $u$  as `U(t,u0,k,omega)` for use in `maple`.

**Steady-state solution.** The steady-state solution  $u_{SS}$  is obtained from the general formula  $u = u_h + u_p$  by dropping all terms containing a negative exponential. It depends on  $t$ ,  $k$  and  $\omega$  but it is independent of  $u_0$ .

### Problem 1.1. (Solution formulas for $u_p$ and $u$ )

Derive by hand, using integral tables, an explicit symbolic formula for  $u_p(t)$ . Display a final formula for  $u = u_h + u_p$  which depends only on  $t$ ,  $u_0$ ,  $k$  and  $\omega$ . Check your hand answer for  $u$  in `maple`. The only `maple` assist in this problem is the answer check.

### Problem 1.2. (Steady-state Periodic Solution)

Derive by hand a formula for the steady-state periodic solution  $u_{SS}$  of  $u' + ku = kA(t)$ . The only `maple` assist in this problem is an answer check.

### Problem 1.3. (Ambient Temperature Plot)

Plot  $A(t)$  on  $0 \leq t \leq 24$  using `maple`. Pencil-in coordinates for the maxima and minima.

### Problem 1.4. (Indoor Temperature $u$ )

Use `maple` to make one graphic with four plots of  $u$  versus  $t$ ,  $0 \leq t \leq 72$  hours. Use  $\omega = \pi/12$  and specialized value  $k = 0.3$  for midnight temperatures  $u_0 = 40$ ,  $u_0 = 50$ ,  $u_0 = 60$ ,  $u_0 = 70$ . Hand-label the maxima and minima.

### Problem 1.5. (Indoor-Outdoor Variation)

Compare in a `maple` graphic the indoor and outdoor temperature oscillations over a 48-hour period assuming  $k = 0.3$ ,  $u_0 = 69$ ,  $\omega = \pi/12$ . Compute the indoor temperature variation from this plot. Find the phase delay using a second plot of steady-state and outdoor temperatures.

### Problem 1.6. (Freezing Pipes)

Assume  $\omega = \pi/12$  and the insulation constant  $k$  ranges from 0.1 to 0.5. Suppose the inside temperature is 69 degrees at midnight when the furnace is turned off. Report approximate ranges of hours during the first 72 hours, for which the indoor temperature is at or below 30 degrees. Justify your logic used to find the ranges, in a short paragraph. Illustrate with a computer graphic.

#### Problem notes.

**Notes on 1.1:** The integration problem to be solved by hand using the book's integral tables is

$$u_p(t) = ke^{-kt} \int_0^t e^{kx} (35 - 15 \cos \omega(x-3)) dx.$$

A change of variable  $u = x - 3$  allows use of integral table entry  $\int e^{au} \cos bu \, du$ . The symbol  $\omega$  will be set to  $\pi/12$ , but for simplicity, use symbol  $\omega$  throughout.

The formulae for  $u_h$  and  $u_p$  are used again in 1.2 in order to derive the steady-state solution. Your answer *must contain* symbols `t, u0, k, omega`.

The answer check in `maple` is organized as follows. The complications of setting  $\omega = \pi/12$  are avoided here by leaving  $\omega$  as a symbol, since it does not affect the answer check.

```
# Test LHS=RHS for u'+ku=kA.
t:='t': u0:='u0': omega:='omega': k:='k':
myANS:=your hand-derived formula for u=uh+up:
LHS:=diff(myANS,t)+k*myANS:
RHS:=k*(35-15*cos(omega*(t-3))):
simplify(expand(LHS-RHS));
```

A successful test of  $LHS = RHS$  produces answer *zero*, or an expression that reduces to zero.

The algebra system `maple` can verify table use, for example, `int(exp(k*x)*cos(w*(x-3)),x=0..t)` will check some of the hand integration details.

**Notes on 1.2:** The steady-state solution is derived from the hand-generated symbolic solution  $u = u_h + u_p$  in 1.1 by dropping all terms that contain  $e^{-kt}$ . The answer, where  $\omega_0 = \pi/12$ :

$$u_{ss} = 35 - \frac{15k}{k^2 + \omega_0^2} (k \cos \omega_0(t - 3) + \omega_0 \sin \omega_0(t - 3)).$$

To check your answer, use `maple` as in 1.1.

To get `maple` to report the above formula, it is essential to evaluate everything with  $\omega$  as a *symbol*, to wit, use in `maple` the statement `omega:='omega':`. Beware of writing `omega:=Pi/12`, which defines  $\omega$  to be a constant, unless you undo the effect immediately afterward.

**Notes on 1.3:** The maximum and minimum of  $A(t)$  occur when the cosine term is smallest and largest, which is when the argument of the cosine is  $\pi$  or 0, respectively. To plot, use `plot(AA(t,Pi/12),t=0..24);` in `maple`, with `AA` defined by `AA:=(t,omega)->35-15*cos(omega*(t-3));`.

**Notes on 1.4:** The solution  $u = U(t, u_0, k, \omega)$  obtained in 1.1 can be programmed in `maple` as follows:

```
t:='t':u0:='u0':k:='k':omega:='omega':
U:=(t,u0,k,omega)-> your answer of u=uh+up from 1.1:
```

For the `maple` plot, use the following example, which puts two of the four plots onto one graphic. Beware: `pi` is a variable but `Pi` is the constant 3.14159: *case* is significant in `maple`.

```
t:='t': u0:='u0': omega:='omega': k:='k':      # Example
U:=(t,u0,k,omega)->u0*exp(-k*t)*cos(omega*t): # Not the 1.1 answer!
u1:=U(t,40,0.3,Pi/12): u2:=U(t,50,0.3,Pi/12):
plot({u1,u2},t=0..72);
```

Missing curves? Probably, the missing curves are defined to contain an unevaluated variable name, like `pi` instead of `Pi`. Curves vanish on the printer? Add `color=black` to the plot command.

**Notes on 1.5:** The two plots are placed onto one graphic by this `maple` command:

```
plot({U(t,69,0.3,Pi/12),AA(t,Pi/12)},t=0..48);
```

Click with the left mouse button on the high and low spots in the graphic. Somewhere on the `maple` worksheet the coordinates of the click are displayed, and this is enough to find a good approximation to the max and min values.

The indoor temperature variation is just the maximum minus the minimum, as computed from the first plot. Beware:  $u$  has a maximum at  $t = 0$ .

The phase shift is found from a second plot, which uses the steady-state solution  $u_{ss}(t)$  obtained in 1.2:

```
plot({uss(t,0.3,Pi/12),AA(t,Pi/12)},t=0..48);
```

The shift is computed as  $|T_2 - T_1|$ , where  $A(T_1) = \max A(t)$  and  $u_{ss}(T_2) = \max u_{ss}(t)$ . Look at the graphic to find sane answers for  $T_1$  and  $T_2$ . See the textbook for a more complete discussion of the ideas. Beware:  $T_1$  and  $T_2$  are less than 20.

**Notes on 1.6:** A computer algebra assist for this problem can be found in `maple`'s function `implicitplot`. This function can plot the equation  $u(t, 69, k, \pi/12) = 30$  over the domain  $0 \leq t \leq 72$ ,  $0.2 \leq k \leq 0.5$ . From this plot, and the 3D-plot  $z = u(x, 69, y, \pi/12)$ , the question is easily answered.

```
with(plots):
t:='t':u0:='u0':k:='k':omega:='omega':
U:=(t,u0,k,omega)-> your answer of uh+up from 1.1:
implicitplot(U(t,69,k,Pi/12)=30,t=0..72,k=0.2..0.5);
plot3d({U(t,69,k,Pi/12),30},t=0..72,k=0.2..0.5);
```

Zoom in on the implicit plot by using a smaller time domain, suggested by the larger plot. The relation between the implicit plot and the 3D plot is seen by slicing the 3D plot at height  $z = 30$  to obtain a “bread slice,” depicted in the implicit plot as that “bread slice” projected into the  $xy$ -plane.

Physically, inside temperature 30 degrees is reached a few hours *after* the outside temperature drops below 30 degrees. During 72 hours, there are three such inside temperature drops (see the 3D-plot, where  $z$ =temperature). For  $k < 0.25$ , inside temperature 30F is not reached in the first 23 hours. Please report an answer like `9-12, 37-39, 56-59 hours` plus an explanation of the logic applied to obtain this answer from the graphics.