2.7 Logistic Equation

The 1845 work of Belgian demographer and mathematician Pierre François Verhulst (1804–1849) modified the classical growth-decay equation \( y' = ky \), replacing \( k \) by \( a - by \), to obtain the **logistic equation**

\[
y' = (a - by)y. \tag{1}
\]

The solution of the logistic equation (1) is (details on page 11)

\[
y(t) = \frac{ay(0)}{by(0) + (a - by(0))e^{-at}}. \tag{2}
\]

The logistic equation (1) applies not only to human populations but also to populations of fish, animals and plants, such as yeast, mushrooms or wildflowers. The \( y \)-dependent growth rate \( k = a - by \) allows the model to have a finite **limiting population** \( a/b \). The constant \( M = a/b \) is called the **carrying capacity** by demographers. Verhulst introduced the terminology **logistic curves** for the solutions of (1).

To use the Verhulst model, a demographer must supply three population counts at three different times; these values determine the constants \( a, b \) and \( y(0) \) in solution (2).

Logistic Models

Below are some variants of the basic logistic model known to researchers in medicine, biology and ecology.

**Limited Environment.** A container of \( y(t) \) flies has a **carrying capacity** of \( N \) insects. A growth-decay model \( y' = Ky \) with combined growth-death rate \( K = k(N - y) \) gives the model \( y' = k(N - y)y \).

**Spread of a Disease.** The initial size of the susceptible population is \( N \). Then \( y \) and \( N - y \) are the number of infectives and susceptibles. Chance encounters spread the incurable disease at a rate proportional to the infectives and the susceptibles. The model is \( y' = ky(N - y) \). The spread of rumors has an identical model.

**Explosion–Extinction.** The number \( y(t) \) of alligators in a swamp can satisfy \( y' = Ky \) where the growth-decay constant \( K \) is proportional to \( y - M \) and \( M \) is a **threshold population**. The logistic model \( y' = k(y - M)y \) gives **extinction** for initial populations smaller than \( M \) and a **doomsday population** explosion \( y(t) \to \infty \) for initial populations greater than \( M \). This model ignores harvesting.
2.7 Logistic Equation

**Constant Harvesting.** The number \( y(t) \) of fish in a lake can satisfy a logistic model \( y' = (a - by) - h \), provided fish are harvested at a constant rate \( h > 0 \). This model can be written as \( y' = k(N - y)(y - M) \) for small harvesting rates \( h \), where \( N \) is the **carrying capacity** and \( M \) is the **threshold population**.

**Variable Harvesting.** The special logistic model \( y' = (a - by) - hy \) results by harvesting at a non-constant rate proportional to the present population \( y \). The effect is to decrease the natural growth rate \( a \) by the constant amount \( h \) in the standard logistic model.

**Restocking.** The equation \( y' = (a - by) - h \sin(\omega t) \) models a logistic population that is periodically harvested and restocked with maximal rate \( h > 0 \). The period is \( T = 2\pi/\omega \). The equation might model extinction for stocks less than some threshold population \( y_0 \), and otherwise a stable population that oscillates about an ideal carrying capacity \( a/b \) with period \( T \).

29 Example (Limited Environment) Find the equilibrium solutions and the carrying capacity for the logistic equation \( P' = 0.04(2 - 3P)P \). Then solve the equation.

**Solution:** The given differential equation can be written as the separable autonomous equation \( P' = G(P) \) where \( G(y) = 0.04(2 - 3P)P \). Equilibria are obtained as \( P = 0 \) and \( P = 2/3 \), by solving the equation \( G(P) = 0.04(2 - 3P)P = 0 \). The carrying capacity is the stable equilibrium \( P = 2/3 \); here we used the derivative \( G'(P) = 0.04(2 - 6P) \) and evaluations \( G'(0) > 0, G'(2/3) < 0 \) to determine that \( P = 2/3 \) is a stable sink or funnel.

30 Example (Spread of a Disease) In each model, find the number of infectives and the number of susceptibles at \( t = 10 \) for the model \( y' = 2(5 - 3P)y \), \( y(0) = 1 \).

**Solution:** Write the differential equation in the form \( y' = 6(5/3 - P)P \) and then identify \( k = 6, N = 5/3 \). We will determine the number of infectives \( y(10) \) and the number of susceptibles \( N - y(10) \).

Using recipe (2) with \( a = 10, b = 6 \) and \( y(0) = 1 \) gives

\[
y(t) = \frac{10}{6 + 4e^{-10t}}.
\]

Then the number of infectives is \( y(10) \approx 10/6 \), which is the carrying capacity \( N = 5/3 \), and the number of susceptibles is \( N - y(10) \approx 0 \).

31 Example (Explosion-Extinction) Classify the model as explosion or extinction: \( y' = 2(y - 100)y \), \( y(0) = 200 \).
**Solution:** Let \( G(y) = 2(y - 100)y \), then \( G(y) = 0 \) exactly for equilibria \( y = 100 \) and \( y = 0 \), at which \( G'(y) = 4y - 200 \) satisfies \( G'(200) > 0 \), \( G'(0) < 0 \). The initial value \( y(0) = 200 \) is above the equilibrium \( y = 100 \). Because \( y = 100 \) is a source, then \( y \to \infty \), which implies the model is explosion.

A second, direct analysis can be made from the differential equation \( y' = 2(y - 100)y \): \( y'(0) = 2(200 - 100)200 > 0 \) means \( y \) increases from 200, causing \( y \to \infty \) and explosion.

32 **Example (Constant Harvesting)** Find the carrying capacity \( N \) and the threshold population \( M \) for the harvesting equation \( P' = (3 - 2P)P - 1 \).

**Solution:** Solve the equation \( G(P) = 0 \) where \( G(P) = (3 - 2P)P - 1 \). The answers \( P = 1/2, P = 1 \) imply that \( G(P) = -2(P - 1)(P - 1/2) = (1 - 2P)(P - 1) \). Comparing to \( P' = k(N - P)(P - M) \), then \( N = 1/2 \) is the carrying capacity and \( M = 1 \) is the threshold population.

33 **Example (Variable Harvesting)** Re-model the variable harvesting equation \( P' = (3 - 2P)P - P \) as \( y' = (a - by)y \) and solve the equation by recipe (2), page 126.

**Solution:** The equation is rewritten as \( P' = 2P - 2P^2 = (2 - 2P)P \). This has the form of \( y' = (a - by)y \) where \( a = b = 2 \). Then (2) implies

\[
P(t) = \frac{2P_0}{2P_0 + (2 - 2P_0)e^{-2t}}
\]

which simplifies to

\[
P(t) = \frac{P_0}{P_0 + (1 - P_0)e^{-2t}}.
\]

34 **Example (Restocking)** Make a direction field graphic by computer for the restocking equation \( P' = (1 - P)P - 2 \sin(2\pi t) \). Using the graphic, report (a) an estimate for the carrying capacity \( C \) and (b) approximations for the amplitude \( A \) and period \( T \) of a periodic solution which oscillates about \( P = C \).

**Solution:** The computer algebra system maple is used with the code below to make Figure 5. An essential feature of the maple code is plotting of multiple solution curves. For instance, \([P(0)=1.3]\) in the ics of initial conditions causes the solution to the problem \( P' = (1 - P)P - 2 \sin(2\pi t), P(0) = 1.3 \) to be added to the graphic.

The resulting graphic, which contains 13 solution curves, shows that all solution curves limit as \( t \to \infty \) to what appears to be a unique periodic solution.

Using features of the maple interface, it is possible to click the mouse and determine estimates for the maxima \( M = 1.26 \) and minima \( m = 0.64 \) of the apparent periodic solution, obtained by experiment. Then (a) \( C = (M + m)/2 = 0.95 \), (b) \( A = (M - m)/2 = 0.31 \) and \( T = 1 \). The experimentally obtained period \( T = 1 \) matches the period of the term \(-2\sin(2\pi t)\).
with(DEtools):
de:=diff(P(t),t)=(1-P(t))*P(t)-2*sin(2*Pi* t);
ics:=[[P(0)=1.4],[P(0)=1.3],[P(0)=1.2],[P(0)=1.1],[P(0)=0.1],
    [P(0)=0.2],[P(0)=0.3],[P(0)=0.4],[P(0)=0.5],[P(0)=0.6],
    [P(0)=0.7],[P(0)=0.8],[P(0)=0.9]];
opts:=stepsize=0.05,arrows=none:
DEplot(de,P(t),t=-3..12,P=-0.1..1.5,ics,opts);

Figure 5. Solutions of $P' = (1 - P)P - 2 \sin(2\pi t)$.
The maximum is 1.26.
The minimum is 0.64.
Oscillation is about the line $P = 0.95$ with period 1.

Exercises 2.7

Limited Environment. Find the equilibrium solutions and the carrying capacity for each logistic equation.
1. $P' = 0.01(2 - 3P)P$
2. $P' = 0.2P - 3.5P^2$
3. $y' = 0.01(-3 - 2y)y$
4. $y' = -0.3y - 4y^2$
5. $u' = 30u + 4u^2$
6. $u' = 10u + 3u^2$
7. $w' = 2(2 - 5w)w$
8. $w' = -2(3 - 7w)w$
9. $Q' = Q^2 - 3(Q - 2)Q$
10. $Q' = -Q^2 - 2(Q - 3)Q$

Spread of a Disease. In each model, find the number of susceptibles and then the number of infectives at $t = 0.557$. Follow Example 30, page 127. A calculator is required for approximations.
11. $y' = (5 - 3P)y$, $y(0) = 1$
12. $y' = (13 - 3y)y$, $y(0) = 2$
13. $y' = (5 - 12y)y$, $y(0) = 2$
14. $y' = (15 - 4y)y$, $y(0) = 10$
15. $P' = (2 - 3P)P$, $P(0) = 500$
16. $P' = (5 - 3P)P$, $P(0) = 200$
17. $P' = 2P - 5P^2$, $P(0) = 100$
18. $P' = 3P - 8P^2$, $P(0) = 10$
19. $y' = 2(y - 100)y$, $y(0) = 200$
20. $y' = 2(y - 200)y$, $y(0) = 300$
21. $y' = -100y + 250y^2$, $y(0) = 200$
22. $y' = -50y + 3y^2$, $y(0) = 25$
23. $y' = -60y + 70y^2$, $y(0) = 30$
24. $y' = -540y + 70y^2$, $y(0) = 30$
25. $y' = -16y + 12y^2$, $y(0) = 1$
26. $y' = -8y + 12y^2$, $y(0) = 1/2$
Constant Harvesting. Find the carrying capacity $N$ and the threshold population $M$.

27. $P' = (3 - 2P)P - 1$
28. $P' = (4 - 3P)P - 1$
29. $P' = (5 - 4P)P - 1$
30. $P' = (6 - 5P)P - 1$
31. $P' = (6 - 3P)P - 1$
32. $P' = (6 - 4P)P - 1$
33. $P' = (8 - 5P)P - 2$
34. $P' = (8 - 3P)P - 2$
35. $P' = (9 - 4P)P - 2$
36. $P' = (10 - P)P - 2$

Variable Harvesting. Re-model the variable harvesting equation as $y' = (a - by)y$ and solve the equation by recipe (2), page 126.

37. $P' = (3 - 2P)P - P$
38. $P' = (4 - 3P)P - P$
39. $P' = (5 - 4P)P - P$
40. $P' = (6 - 5P)P - P$
41. $P' = (6 - 3P)P - P$
42. $P' = (6 - 4P)P - P$
43. $P' = (8 - 5P)P - 2P$
44. $P' = (8 - 3P)P - 2P$
45. $P' = (9 - 4P)P - 2P$
46. $P' = (10 - P)P - 2P$

Restocking. Make a direction field graphic by computer, following Example 34. Using the graphic, report (a) an estimate for the carrying capacity $C$ and (b) approximations for the amplitude $A$ and period $T$ of a periodic solution which oscillates about $y = C$.

47. $P' = (1 - P)P - \sin(5\pi t)$
48. $P' = (1 - P)P - 1.5\sin(5\pi t)$
49. $P' = (2 - P)P - 3\sin(7\pi t)$
50. $P' = (2 - P)P - \sin(7\pi t)$
51. $P' = (4 - 3P)P - 2\sin(3\pi t)$
52. $P' = (4 - 2P)P - 3\sin(3\pi t)$
53. $P' = (10 - 9P)P - 3\sin(4\pi t)$
54. $P' = (10 - 9P)P - \sin(4\pi t)$
55. $P' = (5 - 4P)P - 2\sin(8\pi t)$
56. $P' = (5 - 4P)P - 3\sin(8\pi t)$