2.7 Logistic Equation

The 1845 work of Belgian demographer and mathematician Pierre Francois Verhulst (1804–1849) modified the classical growth-decay equation $y' = ky$, replacing k by $a - by$, to obtain the **logistic equation**

$$y' = (a - by)y. \tag{1}$$

The solution of the logistic equation (1) is (details on page 11)

$$y(t) = \frac{ay(0)}{by(0) + (a - by(0))e^{-at}}. \tag{2}$$

The logistic equation (1) applies not only to human populations but also to populations of fish, animals and plants, such as yeast, mushrooms or wildflowers. The y-dependent growth rate $k = a - by$ allows the model to have a finite *limiting population* a/b. The constant $M = a/b$ is called the **carrying capacity** by demographers. Verhulst introduced the terminology *logistic curves* for the solutions of (1).

To use the Verhulst model, a demographer must supply three population counts at three different times; these values determine the constants a, b and $y(0)$ in solution (2).

Logistic Models

Below are some variants of the basic logistic model known to researchers in medicine, biology and ecology.

Limited Environment. A container of $y(t)$ flies has a *carrying capacity* of N insects. A growth-decay model $y' = Ky$ with combined growth-death rate $K = k(N - y)$ gives the model $y' = k(N - y)y$.

Spread of a Disease. The initial size of the susceptible population is N. Then y and $N - y$ are the number of infectives and susceptibles. Chance encounters spread the incurable disease at a rate proportional to the infectives and the susceptibles. The model is $y' = ky(N - y)$. The spread of rumors has an identical model.

Explosion–Extinction. The number $y(t)$ of alligators in a swamp can satisfy $y' = Ky$ where the growth-decay constant K is proportional to $y - M$ and M is a **threshold population**. The logistic model $y' = k(y - M)y$ gives **extinction** for initial populations smaller than M and a *doomsday* population **explosion** $y(t) \to \infty$ for initial populations greater than M. This model ignores harvesting.

Constant Harvesting. The number $y(t)$ of fish in a lake can satisfy a logistic model $y' = (a - by)y - h$, provided fish are **harvested** at a constant rate $h > 0$. This model can be written as $y' = k(N - y)(y - M)$ for small harvesting rates h, where N is the *carrying capacity* and M is the *threshold population.*

Variable Harvesting. The special logistic model $y' = (a - by)y - hy$ results by **harvesting** at a non-constant rate proportional to the present population y. The effect is to decrease the natural growth rate a by the constant amount h in the standard logistic model.

Restocking. The equation $y' = (a - by)y - h\sin(\omega t)$ models a logistic population that is periodically harvested and restocked with maximal rate $h > 0$. The period is $T = 2\pi/\omega$. The equation might model extinction for stocks less than some threshold population y_0, and otherwise a stable population that oscillates about an ideal carrying capacity a/b with period T.

29 Example (Limited Environment) Find the equilibrium solutions and the carrying capacity for the logistic equation $P' = 0.04(2 - 3P)P$. Then solve the equation.

Solution: The given differential equation can be written as the separable autonomous equation $P' = G(P)$ where $G(y) = 0.04(2-3P)P$. Equilibria are obtained as $P = 0$ and $P = 2/3$, by solving the equation $G(P) = 0.04(2-3P)P = 0$. The carrying capacity is the stable equilibrium $P = 2/3$; here we used the derivative $G'(P) = 0.04(2 - 6P)$ and evaluations $G'(0) > 0$, $G'(2/3) < 0$ to determine that $P = 2/3$ is a stable sink or funnel.

30 Example (Spread of a Disease) In each model, find the number of infectives and the number of susceptibles at $t = 10$ for the model $y' = 2(5-3P)y$, $y(0) = 1$.

Solution: Write the differential equation in the form $y' = 6(5/3 - P)P$ and then identify $k = 6$, $N = 5/3$. We will determine the number of infectives $y(10)$ and the number of susceptibles $N - y(10)$.

Using recipe (2) with $a = 10$, $b = 6$ and $y(0) = 1$ gives

$$y(t) = \frac{10}{6 + 4e^{-10t}}.$$

Then the number of infectives is $y(10) \approx 10/6$, which is the carrying capacity $N = 5/3$, and the number of susceptibles is $N - y(10) \approx 0$.

31 Example (Explosion-Extinction) Classify the model as **explosion** or **extinction**: $y' = 2(y - 100)y$, $y(0) = 200$.

Solution: Let $G(y) = 2(y-100)y$, then $G(y) = 0$ exactly for equilibria $y = 100$ and $y = 0$, at which $G'(y) = 4y - 200$ satisfies $G'(200) > 0$, $G'(0) < 0$. The initial value $y(0) = 200$ is above the equilibrium $y = 100$. Because $y = 100$ is a source, then $y \to \infty$, which implies the model is **explosion**.

A second, direct analysis can be made from the differential equation $y' = 2(y-100)y$: $y'(0) = 2(200-100)200 > 0$ means y increases from 200, causing $y \to \infty$ and explosion.

32 **Example (Constant Harvesting)** Find the carrying capacity N and the threshold population M for the harvesting equation $P' = (3 - 2P)P - 1$.

Solution: Solve the equation $G(P) = 0$ where $G(P) = (3 - 2P)P - 1$. The answers $P = 1/2$, $P = 1$ imply that $G(P) = -2(P-1)(P-1/2) = (1-2P)(P-1)$. Comparing to $P' = k(N-P)(P-M)$, then $N = 1/2$ is the carrying capacity and $M = 1$ is the threshold population.

33 **Example (Variable Harvesting)** Re-model the variable harvesting equation $P' = (3-2P)P - P$ as $y' = (a-by)y$ and solve the equation by recipe (2), page 126.

Solution: The equation is rewritten as $P' = 2P - 2P^2 = (2 - 2P)P$. This has the form of $y' = (a - by)y$ where $a = b = 2$. Then (2) implies

$$P(t) = \frac{2P_0}{2P_0 + (2 - 2P_0)e^{-2t}}$$

which simplifies to

$$P(t) = \frac{P_0}{P_0 + (1 - P_0)e^{-2t}}.$$

34 **Example (Restocking)** Make a direction field graphic by computer for the restocking equation $P' = (1 - P)P - 2\sin(2\pi t)$. Using the graphic, report (a) an estimate for the carrying capacity C and (b) approximations for the amplitude A and period T of a periodic solution which oscillates about $P = C$.

Solution: The computer algebra system `maple` is used with the code below to make Figure 5. An essential feature of the `maple` code is plotting of multiple solution curves. For instance, `[P(0)=1.3]` in the list `ics` of initial conditions causes the solution to the problem $P' = (1 - P)P - 2\sin(2\pi t)$, $P(0) = 1.3$ to be added to the graphic.

The resulting graphic, which contains 13 solution curves, shows that all solution curves limit as $t \to \infty$ to what appears to be a unique periodic solution.

Using features of the `maple` interface, it is possible to click the mouse and determine estimates for the maxima $M = 1.26$ and minima $m = 0.64$ of the apparent periodic solution, obtained by experiment. Then (a) $C = (M+m)/2 = 0.95$, (b) $A = (M-m)/2 = 0.31$ and $T = 1$. The experimentally obtained period $T = 1$ matches the period of the term $-2\sin(2\pi t)$.

```
with(DEtools):
de:=diff(P(t),t)=(1-P(t))*P(t)-2*sin(2*Pi* t);
ics:=[[P(0)=1.4],[P(0)=1.3],[P(0)=1.2],[P(0)=1.1],[P(0)=0.1],
[P(0)=0.2],[P(0)=0.3],[P(0)=0.4],[P(0)=0.5],[P(0)=0.6],
[P(0)=0.7],[P(0)=0.8],[P(0)=0.9]];
opts:=stepsize=0.05,arrows=none:
DEplot(de,P(t),t=-3..12,P=-0.1..1.5,ics,opts);
```

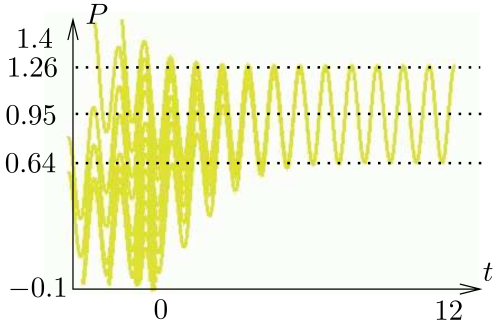

Figure 5. Solutions of $P' = (1-P)P - 2\sin(2\pi t)$**.**

The maximum is 1.26.
The minimum is 0.64.
Oscillation is about the line $P = 0.95$ with period 1.

Exercises 2.7

Limited Environment. Find the equilibrium solutions and the carrying capacity for each logistic equation.

1. $P' = 0.01(2-3P)P$
2. $P' = 0.2P - 3.5P^2$
3. $y' = 0.01(-3-2y)y$
4. $y' = -0.3y - 4y^2$
5. $u' = 30u + 4u^2$
6. $u' = 10u + 3u^2$
7. $w' = 2(2-5w)w$
8. $w' = -2(3-7w)w$
9. $Q' = Q^2 - 3(Q-2)Q$
10. $Q' = -Q^2 - 2(Q-3)Q$

Spread of a Disease. In each model, find the number of susceptibles and then the number of infectives at $t = 0.557$. Follow Example 30, page 127. A calculator is required for approximations.

11. $y' =' (5-3P)y$, $y(0) = 1$.
12. $y' = (13-3y)y$, $y(0) = 2$.
13. $y' = (5-12y)y$, $y(0) = 2$.
14. $y' = (15-4y)y$, $y(0) = 10$.
15. $P' = (2-3P)P$, $P(0) = 500$.
16. $P' = (5-3P)P$, $P(0) = 200$.
17. $P' = 2P - 5P^2$, $P(0) = 100$.
18. $P' = 3P - 8P^2$, $P(0) = 10$.

Explosion–Extinction. Classify the model as **explosion** or **extinction**.

19. $y' = 2(y-100)y$, $y(0) = 200$
20. $y' = 2(y-200)y$, $y(0) = 300$
21. $y' = -100y + 250y^2$, $y(0) = 200$
22. $y' = -50y + 3y^2$, $y(0) = 25$
23. $y' = -60y + 70y^2$, $y(0) = 30$
24. $y' = -540y + 70y^2$, $y(0) = 30$
25. $y' = -16y + 12y^2$, $y(0) = 1$
26. $y' = -8y + 12y^2$, $y(0) = 1/2$

Constant Harvesting. Find the carrying capacity N and the threshold population M.

27. $P' = (3-2P)P - 1$

28. $P' = (4-3P)P - 1$

29. $P' = (5-4P)P - 1$

30. $P' = (6-5P)P - 1$

31. $P' = (6-3P)P - 1$

32. $P' = (6-4P)P - 1$

33. $P' = (8-5P)P - 2$

34. $P' = (8-3P)P - 2$

35. $P' = (9-4P)P - 2$

36. $P' = (10-P)P - 2$

Variable Harvesting. Re-model the variable harvesting equation as $y' = (a-by)y$ and solve the equation by recipe (2), page 126.

37. $P' = (3-2P)P - P$

38. $P' = (4-3P)P - P$

39. $P' = (5-4P)P - P$

40. $P' = (6-5P)P - P$

41. $P' = (6-3P)P - P$

42. $P' = (6-4P)P - P$

43. $P' = (8-5P)P - 2P$

44. $P' = (8-3P)P - 2P$

45. $P' = (9-4P)P - 2P$

46. $P' = (10-P)P - 2P$

Restocking. Make a direction field graphic by computer, following Example 34. Using the graphic, report (a) an estimate for the carrying capacity C and (b) approximations for the amplitude A and period T of a periodic solution which oscillates about $y = C$.

47. $P' = (1-P)P - \sin(5\pi t)$

48. $P' = (1-P)P - 1.5\sin(5\pi t)$

49. $P' = (2-P)P - 3\sin(7\pi t)$

50. $P' = (2-P)P - \sin(7\pi t)$

51. $P' = (4-3P)P - 2\sin(3\pi t)$

52. $P' = (4-2P)P - 3\sin(3\pi t)$

53. $P' = (10-9P)P - 3\sin(4\pi t)$

54. $P' = (10-9P)P - \sin(4\pi t)$

55. $P' = (5-4P)P - 2\sin(8\pi t)$

56. $P' = (5-4P)P - 3\sin(8\pi t)$