Math 252
Applied Linear Algebra
Problem Notes on Take-Home Exams I to IX

1. (2×2 **Linear System**) Find all solutions to $x_1 - 5x_2 = 0$, $-x_1 + 5x_2 = 0$. Express the answer in one of the three possible parametric forms:

A *point.* This represents the intersection of two lines.

A *line.* The parametric form is
$x = c_1 + td_1$,
$y = c_2 + td_2$,
with $-\infty < t < \infty$.

A *plane.* In this case, all points (x, y) are answers.

Solution to 1. To explain geometrically the expected kind of answer, observe that $-x_1 + 5x_2 = 0$ is the same as the first equation $x_1 - 5x_2 = 0$, therefore there are not two equations, but only one! The set of planar points satisfying the two equations is exactly the set of points on the straight line $x_1 - 5x_2 = 0$ (an infinite number of points). The **standard form of the solution** is obtained by solving for x_1 in terms of x_2, e.g., $x_1 = 5x_2$, then write out the vector solution X as follows:

$$X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 5x_2 \\ x_2 \end{pmatrix} = x_2 \begin{pmatrix} 5 \\ 1 \end{pmatrix}$$

To each value of x_2 corresponds a solution of the system of equations, i.e., there are infinitely many solutions, represented goemetrically as a line.

1a. (2×4 **Linear System**) Find all solutions to $x_1 - x_2 + 7x_3 - x_4 = 0$, $2x_1 + 3x_2 - 8x_3 + x_4 = 0$.

Solution to 1a. Subtract two times the first equation from the second to get $5x_2 - 22x_3 + 3x_4 = 0$. Divide the new equation by 5 to get $x_2 - \frac{22}{5}x_3 + \frac{3}{5}x_4 = 0$. Keep this as the replacement for the second equation. Add it to the first equation to get its replacement $x_1 + \frac{13}{5}x_3 - \frac{2}{5}x_4 = 0$. The replacement equations are therefore

$$x_1 + \frac{13}{5}x_3 - \frac{2}{5}x_4 = 0,$$

$$x_2 - \frac{22}{5}x_3 + \frac{3}{5}x_4 = 0,$$

which correspond exactly to the *reduced row echelon form* of the system. The *general solution* is

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = x_3 \begin{pmatrix} -\frac{13}{5} \\ \frac{22}{5} \\ 1 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} \frac{2}{5} \\ -\frac{3}{5} \\ 0 \\ 1 \end{pmatrix}.$$

2. (4×4 **Linear System**) Find a fraction-free Gauss-elimination form and the reduced row-echelon form for the following equations: $x_1 - 2x_2 + x_3 + x_4 = 2$, $3x_1 + 2x_3 - 2x_4 = -8$, $4x_2 - x_3 - x_4 = 1$, $5x_1 + 3x_3 - x_4 = 0$.

Solution to 2. A fraction-free Gauss-elimination form can be obtained from the reduced row echelon form by multiplying each row by a suitable factor, to clear the fractions. In reality, there are infinitely many fraction-free forms, so there is no way to give an answer that everyone will arrive at.

It turns out that the reduced row echelon form is also fraction-free, so it can be reported as the fraction-free answer!

The reduced row-echelon form is obtained from the augmented matrix by row operations, using the basic pivot algorithm. The answer for *both questions*:

$$\text{rref} = \left(\begin{array}{cccr|c} 1 & 0 & 0 & 4 & 0 \\ 0 & 1 & 0 & -2 & 0 \\ 0 & 0 & 1 & -7 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right)$$

3. (4 × 3 **Linear System**) Find all solutions to the 4 × 3 system $x_1 + x_2 - x_3 = 0$, $4x_1 - x_2 + 5x_3 = 0$, $-2x_1 + x_2 - 2x_3 = 0$, $3x_1 + 2x_2 - 6x_3 = 0$.

Solution to 3. The augmented matrix and its reduced row echelon form are:

$$\text{aug} = \left(\begin{array}{rrr|c} 1 & 1 & -1 & 0 \\ 4 & -1 & 5 & 0 \\ -1 & 1 & -2 & 0 \\ 3 & 2 & -6 & 0 \end{array}\right), \quad \text{rref} = \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

It follows that $X = 0$ is the only solution.

4. (3 × 4 **Linear System**) Find all solutions to $x_1 - x_2 + x_3 - x_4 = -2$, $-2x_1 + 3x_2 - x_3 + 2x_4 = 5$, $4x_1 - 2x_2 + 2x_3 - 3x_4 = 6$.

Solution to 4. The augmented matrix and its reduced row echelon form are:

$$\text{aug} = \left(\begin{array}{rrrr|r} 1 & -1 & 1 & -1 & -2 \\ -2 & 3 & -1 & 2 & 5 \\ 4 & -2 & 2 & -3 & 6 \end{array}\right), \quad \text{rref} = \left(\begin{array}{cccr|r} \boxed{1} & 0 & 0 & -1/2 & 5 \\ 0 & \boxed{1} & 0 & 1/4 & 4 \\ 0 & 0 & \boxed{1} & -1/4 & -3 \end{array}\right)$$

The **standard form of the solution** is obtained by identifying the **lead variables** and the **arbitrary variables**:

> Variables x_1, x_2 and x_3 are the **lead variables**, because they correspond to a leading 1 in the RREF. See the boxed 1's above.
>
> Variable x_4 is the **arbitrary variable**, because arbitrary variables are the variables left over after removal of the lead variables.

The standard form of the solution X is obtained by replacing each lead variable with its corresponding equation, obtained from the RREF. The arbitrary variables are left untouched. Then:

$$X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} \frac{1}{2}x_4 + 5 \\ \frac{-1}{4}x_4 + 4 \\ \frac{1}{4}x_4 - 3 \\ x_4 \end{pmatrix} = x_4 \begin{pmatrix} \frac{1}{2} \\ \frac{-1}{4} \\ 3 \\ 1 \end{pmatrix} + \begin{pmatrix} 5 \\ 4 \\ -3 \\ 0 \end{pmatrix}$$

4a. (**3 × 4 Linear System**) Use Gauss-Jordan elimination to find the general solution:

$$\begin{array}{rcrcrcrcl} x_1 & + & 2x_2 & + & 4x_3 & - & x_4 & = & 3 \\ 3x_1 & + & 4x_2 & + & 5x_3 & - & x_4 & = & 7 \\ x_1 & + & 3x_2 & + & 4x_3 & + & 5x_4 & = & 4 \end{array}$$

Solution to 4a. The augmented matrix and its reduced row echelon form are:

$$aug = \left(\begin{array}{cccr|c} 1 & 2 & 4 & -1 & 3 \\ 3 & 4 & 5 & -1 & 7 \\ 1 & 3 & 4 & 5 & 4 \end{array}\right), \quad \text{rref} = \left(\begin{array}{cccr|c} \boxed{1} & 0 & 0 & -5 & 1 \\ 0 & \boxed{1} & 0 & 6 & 1 \\ 0 & 0 & \boxed{1} & -2 & 0 \end{array}\right)$$

Variables x_1, x_2 and x_3 are the **lead variables**, because they correspond to a leading 1 in the RREF. See the boxed 1's above.

Variable x_4 is the **arbitrary variable**, because arbitrary variables are the variables left over after removal of the lead variables.

The **standard form of the solution** is obtained by replacing the lead variables x_1, x_2, x_3 by their equations ($x_1 = 5x_4 + 1$, $x_2 = -6x_4 + 1$, $x_3 = 2x_4$), but the arbitrary variable x_4 is untouched. Then:

$$X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 5x_4 + 1 \\ -6x_4 + 1 \\ 2x_4 \\ x_4 \end{pmatrix} = x_4 \begin{pmatrix} 5 \\ -6 \\ 2 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

Important: This *method* for writing out X applies *only* in case the equations are in reduced echelon form. A matrix C is in **reduced row echelon form** provided each nonzero row starts with a **leading** 1, and above and below that leading 1 appear only zeros.

5. (**Linear Combinations**) Compute the result of the linear combination $2u + v - 3w$ where

$$u = \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}, \quad v = \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix}, \quad w = \begin{pmatrix} 9 \\ 1 \\ -4 \end{pmatrix}.$$

Solution to 5. The result of the linear combination is

$$2u + v - 3w = 2\begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix} - 3\begin{pmatrix} 9 \\ 1 \\ -4 \end{pmatrix} = \begin{pmatrix} -25 \\ -1 \\ 10 \end{pmatrix}.$$

5a. (**Equality of Vectors**) Let

$$u = \begin{pmatrix} 1 \\ x \\ -2 \end{pmatrix}, \quad v = \begin{pmatrix} x + 1 \\ 0 \\ 2 \end{pmatrix}, \quad w = \begin{pmatrix} 9 \\ 1 \\ -4x \end{pmatrix}$$

The linear combination $p = 2u + v - 3w$ depends upon x. Is there a value of x such that

$$p = \begin{pmatrix} -22.9 \\ -0.8 \\ 11.2 \end{pmatrix}?$$

Solution to 5a. The linear combination p is given by

$$p = 2\begin{pmatrix} 1 \\ x \\ -2 \end{pmatrix} + \begin{pmatrix} x+1 \\ 0 \\ 2 \end{pmatrix} - 3\begin{pmatrix} 9 \\ 1 \\ -4x \end{pmatrix} = \begin{pmatrix} x-24 \\ 2x-3 \\ 12x-2 \end{pmatrix}.$$

There is a value of x such that

$$p = \begin{pmatrix} -22.9 \\ -0.8 \\ 11.2 \end{pmatrix}$$

exactly when the components agree, i.e., $x - 24 = -22.9$, $2x - 3 = -0.8$, $12x - 3 = 11.2$. This happens for $x = 1.1$ (check all three equations!).

6. **(Largest Linearly Independent Set)** Extract from the list

$$\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

a largest set of linearly independent vectors.

Solution to 6. To extract a largest linearly independent set of vectors from a list v_1, v_2, v_3, $v4$ requires an algorithm be followed.

The algorithm begins by choosing the set to be the single element v_1, already an independent set. Try to add v_2 to the set. This will fail if v_2 is a multiple of v_1. Then try to add v_3 to the set. It will fail if v_3 is a combination of vectors already in the set. Finally, try to add v_4 to the set, which fails if v_4 is a combination of vectors already in the set. When complete, the set is independent by construction and of largest size.

The largest independent set from the list

$$\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

has size 2. The algorithm gives a basis

$$\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}.$$

An effective way to test independence of vectors is to find the RREF of the matrix with these vectors as columns. If the rank is the same as the number of columns, then the set is independent.

6a. **(Vector Spaces)** Extract from the list x, $x-1$, $2x+1$, x^2-x, x^2+x a largest list of linearly independent functions.

Solution to 6a. The list x, $x-1$, $2x+1$, x^2-x, x^2+x can be mapped to a list of column vectors

$$\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

where

$$\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} \quad \leftrightarrow \quad c_1 + c_2 x + c_3 x^2.$$

Then the largest independent list if column vectors corresponds exactly to the largest set of independent polynomials. The RREF method gives the largest list as

$$\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$$

and therefore the largest list of independent polynomials is x, $x - 1$, $x^2 - x$ (the size is 3, there are many independent lists of size 3).

7. (**Basis for a Subspace**) Find a basis for the set of vectors $\vec{v} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ in $\mathbf{R}^3$ that satisfy the equation $5x + 6y - 2z = 0$.

Solution to 7. To find a basis for the set of vectors $X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ in $\mathbf{R}^3$ that satisfy the equation $5x+6y-2z = 0$, find the RREF of the matrix A given above and write out the complete solution X of the linear system $AX = 0$:

$$X = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} (-6/5)y + (2/5)z \\ y \\ z \end{pmatrix} = y \begin{pmatrix} -6/5 \\ 1 \\ 0 \end{pmatrix} + z \begin{pmatrix} 2/5 \\ 0 \\ 1 \end{pmatrix}.$$

A basis can be read off the standard form of the solution X:

$$\begin{pmatrix} -6/5 \\ 1 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 2/5 \\ 0 \\ 1 \end{pmatrix}.$$

7a. (**Subspace Criterion**) Show that the set of vectors $\overrightarrow{v} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ in $\mathbf{R}^3$ that satisfy the equation $5x + 6y - 2z = 0$ is a subspace of $\mathbf{R}^3$.

Solution to 7a. To prove a set is a subspace of $\mathbf{R}^3$ it suffices to establish the conditions in the **Subspace Criterion**:

(a) If X and Y are in the set, then so is $X + Y$.
(b) If X is in the set and k is a constant, then kX is in the set.

A set already known to be a subspace of $\mathbf{R}^3$ has a basis of one, two or three elements (unless the set is the origin). A plane given by an equation $ax + by + cz = 0$ represents a subspace with basis of two elements, because two independent vectors determine a plane. A line through the origin given by a vector equation

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = t \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$$

is a subspace with a basis of one element.

To show that the set of vectors $X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ in $\mathbf{R}^3$ that satisfy the equation $5x + 6y - 2z = 0$ is a subspace of $\mathbf{R}^3$, write the equation as a matrix equation $AX = 0$ where

$$A = \begin{pmatrix} 5 & 6 & -2 \end{pmatrix}.$$

It is routine to check (a) and (b) for the equation $AX = 0$. For example, to check (a), let $AX = 0$ and $BY = 0$, then $A(X + Y) = AX + AY = 0 + 0 = 0$, so $X + Y$ is in the set. Item (b) is similar.

8. **(Rank, Nullity and Nullspace)** Find the rank, nullity and a basis for the null space, given A has rows $[1, -1, 2, 3]$, $[0, 1, 4, 3]$, $[1, 0, 6, 6]$.

Solution to 8. Given A has rows $[1, -1, 2, 3]$, $[0, 1, 4, 3]$, $[1, 0, 6, 6]$, then the reduced row echelon form of the augmented matrix $(A : 0)$ is

$$\text{rref}(A) = \left(\begin{array}{cccc|c} \boxed{1} & 0 & 6 & 6 & 0 \\ 0 & \boxed{1} & 4 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

The rank and nullity are both 2. The standard form of the solution X to the equation $AX = 0$ is obtained using lead variables x_1, x_2 and arbitrary variables x_3, x_4 as follows:

$$X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -6x_3 - 6x_4 \\ -4x_3 - 3x_4 \\ x_3 \\ x_4 \end{pmatrix} = x_3 \begin{pmatrix} -6 \\ -4 \\ 1 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} -6 \\ -3 \\ 0 \\ 1 \end{pmatrix}$$

A basis for the null space can be read off from this answer:

$$\begin{pmatrix} -6 \\ -4 \\ 1 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} -6 \\ -3 \\ 0 \\ 1 \end{pmatrix}.$$

8a. **(Rank, Nullity and Nullspace)** Find the rank, nullity and a basis for the solution space of $AX = 0$, where

$$A = \begin{pmatrix} 1 & 2 & 1 & -1 & 3 \\ 1 & 2 & 2 & 1 & 2 \\ 2 & 4 & 2 & -1 & 7 \end{pmatrix}, \quad X = \begin{pmatrix} x \\ y \\ z \\ u \\ v \end{pmatrix}.$$

Solution to 8a. Rather than form $(A : 0)$, we work with A itself and find the RREF and also the standard form of the solution:

$$A = \begin{pmatrix} 1 & 2 & 1 & -1 & 3 \\ 1 & 2 & 2 & 1 & 2 \\ 2 & 4 & 2 & -1 & 7 \end{pmatrix}, \quad \text{rref}(A) = \begin{pmatrix} \boxed{1} & 2 & 0 & 0 & 7 \\ 0 & 0 & \boxed{1} & 0 & -3 \\ 0 & 0 & 0 & \boxed{1} & 1 \end{pmatrix}.$$

The rank is 3, the nullity 2. The solution is read off from the RREF by observing that the **lead variables** are x, z, u and the **arbitrary variables** are y, v. Then

$$X = \begin{pmatrix} x \\ y \\ z \\ u \\ v \end{pmatrix} = \begin{pmatrix} -2y - 7v \\ y \\ 3v \\ -v \\ v \end{pmatrix} = y \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + v \begin{pmatrix} -7 \\ 0 \\ 3 \\ -1 \\ 1 \end{pmatrix}.$$

A basis for the solution space of $AX = 0$ is given by

$$\begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} -7 \\ 0 \\ 3 \\ -1 \\ 1 \end{pmatrix}.$$

8b. **(Rank, Nullity and Nullspace)** Find the rank, nullity and a basis for the solution space of $AX = 0$, given

$$A = \begin{pmatrix} 1 & -1 & 2 & 3 \\ -2 & 2 & -4 & -6 \\ 2 & -2 & 4 & 6 \\ 3 & -3 & 6 & 9 \end{pmatrix}, \quad X = \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix}.$$

Solution to 8b. Rather than form $(A : 0)$, we work with A itself and find the RREF and also the standard form of the solution:

$$A = \begin{pmatrix} 1 & -1 & 2 & 3 \\ -2 & 2 & -4 & -6 \\ 2 & -2 & 4 & 6 \\ 3 & -3 & 6 & 9 \end{pmatrix}, \quad \text{rref}(A) = \begin{pmatrix} \boxed{1} & -1 & 2 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

The solution is read off from the RREF by observing that the **lead variable** is x and the **arbitrary variables** are y, z, w. Then

$$X = \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} y - 2z - 3w \\ y \\ z \\ w \end{pmatrix} = y \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + z \begin{pmatrix} -2 \\ 0 \\ 1 \\ 0 \end{pmatrix} + w \begin{pmatrix} -3 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

The rank is 1, the nullity is 3 and a basis for the null space is

$$\begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} -2 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} -3 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

Maple notes on problems 2–8b. Linear equations can be entered into `maple` without converting to matrix form. While this form is not convenient for solving the equations, there is a conversion routine called `genmatrix` which creates either the coefficient matrix or the augmented matrix. An example:

```
# Maple V.3 [Augmented matrix errors in V.1, V.2]
eq:=[2*x+3*y-z=0,3*x-4*y+5*z=8,y-z=9];
ans:=[x,y,z];
a:=genmatrix(eq,ans);
#                                  [ 2   3  -1 ]
#                             a:= [ 3  -4   5 ]
#                                  [ 0   1  -1 ]
#
aug:=genmatrix(eq,ans,1);
#  This is what your            [ 2   3  -1   0 ]
#  book calls the         aug:= [ 3  -4   5   8 ]
#  augmented matrix.            [ 0   1  -1   9 ]
```

The fraction–free Gauss-Jordan forms are not unique. The preferred form is given by the `maple` function `ffgausselim`. This form combined with the `maple` command `backsub` can be used to find the solution to a linear system.

The Reduced Row Echelon form (RREF) is unique. In `maple`, the command is called `rref` or `gaussjord`, one being a synonym for the other. From this form the general solution of a linear system can determined by back-substitution using `backsub`.

The RREF method is preferred for most applications done by hand computation. This form identifies the dependent variables as those corresponding to the leading 1's. The other variables will appear as arbitrary constants in the general solution. For example, if the reduced form of an augmented matrix is (leading 1's boxed)

$$\left(\begin{array}{cccc|c} \boxed{1} & -1 & 0 & 0 & 4 \\ 0 & 0 & \boxed{1} & 0 & -5 \\ 0 & 0 & 0 & \boxed{1} & 2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right),$$

then x_1, x_3 and x_4 are dependent variables while x_2 appears in the answer as an arbitrary constant:

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 4 \\ 0 \\ -5 \\ 2 \end{pmatrix} + x_2 \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}.$$

The actual form of the answer in `maple` will contain variable names starting with the letter `t`. The factorization of the answer into basis elements for the kernel is not automatic. Here is an example of how to pass from the output of `gausselim` to the general solution, using `backsub`:

```
with(linalg):
a:=matrix([[1,3,-2,0,2,0],[2,6,-5,-2,4,-3],[0,0,5,10,0,15],[2,6,0,8,4,18]]);
b:=matrix([[0],[-1],[5],[6]]);
c:=augment(a,b);
M:=gausselim(c);
backsub(M);
#
#    [ 1  3  -2   0  2   0 ]        [  0 ]
#    [ 2  6  -5  -2  4  -3 ]        [ -1 ]
#a:= [                     ]   b := [    ]
#    [ 0  0   5  10  0  15 ]        [  5 ]
#    [ 2  6   0   8  4  18 ]        [  6 ]
#
#     [ 1  3  -2   0  2   0   0 ]        [ 1  3  -2   0  2   0   0 ]
#     [ 2  6  -5  -2  4  -3  -1 ]        [ 0  0  -1  -2  0  -3  -1 ]
#c := [                         ]   M := [                         ]
#     [ 0  0   5  10  0  15   5 ]        [ 0  0   0   0  0   6   2 ]
#     [ 2  6   0   8  4  18   6 ]        [ 0  0   0   0  0   0   0 ]
#
#               [ - 3 t2 - 4 t4 - 2 t5,  t2, - 2 t4,  t4,  t5,  1/3 ]
```

The format of the general solution obtained above is not what is normally written in a hand computation. Below is the general solution in the usual hand–written format:

```
#        [ x1 ]   [  0  ]      [ -3 ]      [ -4 ]      [ -2 ]
#        [ x2 ]   [  0  ]      [  1 ]      [  0 ]      [  0 ]
#        [ x3 ]   [  0  ]      [  0 ]      [ -2 ]      [  0 ]
#        [    ] = [     ] + t2 [    ] + t4 [    ] + t5 [    ]
#        [ x4 ]   [  0  ]      [  0 ]      [  1 ]      [  0 ]
#        [ x5 ]   [  0  ]      [  0 ]      [  0 ]      [  1 ]
#        [ x6 ]   [ 1/3 ]      [  0 ]      [  0 ]      [  0 ]
```

A `maple` procedure can be written to display the above general solution. The source file:

`/u/cl/maple/gensol`.

```
# file "gensol"
with(linalg):                                   # Uses linalg package
gensolution:=proc(M)                            # M:=rref(augment(A,b)):
local x,y,n,w,v,s,u,i:
  x:=backsub(M):                                # Solve Ax=b
  n:=coldim(matrix([eval(x)])):                 # Get number of vars
  y:=[seq(x[i],i=1..n)]:                        # Make list of ans
  w:=[seq(t.i,i=1..n)]:                         # Make list of vars
  v:=[seq(x.i,i=1..n)]:                         # Make list of vars
  s:=matrix(n,1,subs(seq(w[i]=0,i=1..n),eval(y)) ): # Particular solution
  for i from 1 to n do
    u:=matrix(n,1,map(diff,eval(y),w[i])):            # basis vector
    if norm(u) <> 0 then s:=eval(s)+w[i]*eval(u): fi: # for variable w[i]
  od:
  s:=matrix(n,1,v)=eval(s):                     # Write out general solution
  RETURN(s):                                    # as an equation.
end:
```

As an example of how to use this procedure consider the following:

```
read gensol:
a:=matrix([[4,-1,2,6],[-1,5,-1,-3],[3,4,1,3] ]);
b:=matrix([[b1],[b2],[b3]]);
c:=augment(a,b);
M:=gausselim(c);
M[3,5]:=0:        # The system is consistent if and only if M[3,5]:=0
gensolution(M);
#
#      [  4  -1   2   6 ]       [ b1 ]        [  4  -1   2   6  b1 ]
# a := [ -1   5  -1  -3 ]  b := [ b2 ]   c := [ -1   5  -1  -3  b2 ]
#      [  3   4   1   3 ]       [ b3 ]        [  3   4   1   3  b3 ]
#
#
#      [ 4   -1     2     6        b1      ]      This system is inconsistent
# M := [ 0  19/4  -1/2  -3/2   b2 + 1/4 b1 ]      unless the last row is all
#      [ 0    0     0     0   b3 - b1 - b2 ]      zeros: b3 - b1 - b2=0.
#
#   [ x1 ]   [ 1/19 b2 + 5/19 b1 ]      [ -9/19 ]      [ -27/19 ]
#   [ x2 ]   [ 4/19 b2 + 1/19 b1 ]      [  2/19 ]      [  6/19  ]
#   [    ] = [                   ] + t3 [       ] + t4 [        ]
#   [ x3 ]   [         0         ]      [   1   ]      [    0   ]
#   [ x4 ]   [         0         ]      [   0   ]      [    1   ]
```

9. **(Inverse Matrix)** Compute the inverse of the matrix whose rows are $[1,0,0]$, $[-2,0,1]$, $[4,6,1]$.

Solution to 9. The inverse of the matrix A whose rows are $[1,0,0]$, $[-2,0,1]$, $[4,6,1]$ is obtained by forming the augmented matrix $B = (A : I)$ and then the RREF of B. Alternatively, it can be computed from the adjugate or adjoint formula, using cofactors of A. The answer:

$$A^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ -1 & -1/6 & 1/6 \\ 2 & 1 & 0 \end{pmatrix}$$

9a. **(Invertible Matrices)** Explain why the matrix whose rows are $[1,0,0]$, $[-2,0,0]$, $[4,6,1]$ is not invertible.

Solution to 9a. The matrix whose rows are $[1,0,0]$, $[-2,0,0]$, $[4,6,1]$ is not invertible, because the rows are dependent (the first and second rows are dependent). Alternatively, the rank is less than the row dimension. Finally, a third way to analyze it comes from the theory of determinants: a matrix is invertible if and only if its determinant is nonzero.

10. **(Inverse of a Matrix)** Calculate A^{-1} if it exists:

$$A = \begin{pmatrix} 1 & 0 & 2 & 3 \\ -1 & 1 & 0 & 4 \\ 2 & 1 & -1 & 3 \\ -1 & 0 & 0 & 0 \end{pmatrix}$$

Solution to 10. The answer, obtained from $\mathrm{rref}((A : I))$:

$$A = \begin{pmatrix} 1 & 0 & 2 & 3 \\ -1 & 1 & 0 & 4 \\ 2 & 1 & -1 & 3 \\ -1 & 0 & 0 & 0 \end{pmatrix}, \quad A^{-1} = \begin{pmatrix} 0 & 0 & 0 & -1 \\ -4 & 9 & -8 & -29 \\ -1 & 3 & -3 & -10 \\ 1 & -2 & 2 & 7 \end{pmatrix}.$$

11. **(Row Space and Column Space)** Find bases for the row space, the column space and the null space of the matrix

$$A = \begin{pmatrix} 4 & 1 & -3 & 5 \\ 2 & 0 & 0 & -2 \\ 6 & 2 & -6 & 12 \end{pmatrix}.$$

Solution to 11. Row reduction to RREF will give a basis for the row space of A. Row reduction for the transpose of A will give a basis for the column space of A.

$$A = \begin{pmatrix} 4 & 1 & -3 & 5 \\ 2 & 0 & 0 & -2 \\ 6 & 2 & -6 & 12 \end{pmatrix}, \quad \mathrm{rref}(A) = \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & -3 & 9 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

$$A^t = \begin{pmatrix} 4 & 2 & 6 \\ 1 & 0 & 2 \\ -3 & 0 & -6 \\ 5 & -2 & 12 \end{pmatrix}, \quad \mathrm{rref}(A^t) = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

The row space is generated by the basis

$$\left(1,0,0,-1 \right), \quad \left(0,1,-3,9 \right).$$

The column space is generated by the basis

$$\begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}.$$

The null space is generated from rref$((A : 0))$ by solving the equation $AX = 0$ for the standard form of the solution X, as follows:

$$\text{rref}((A : 0)) = \left(\begin{array}{cccc|c} \boxed{1} & 0 & 0 & -1 & 0 \\ 0 & \boxed{1} & -3 & 9 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right),$$

$$X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} x_4 \\ 3x_3 - 9x_4 \\ x_3 \\ x_4 \end{pmatrix} = x_3 \begin{pmatrix} 0 \\ 3 \\ 1 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 1 \\ -9 \\ 0 \\ 1 \end{pmatrix}.$$

A basis for the nullspace is therefore

$$\begin{pmatrix} 0 \\ 3 \\ 1 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ -9 \\ 0 \\ 1 \end{pmatrix}.$$

11a. **(Transpose of a Product of Symmetric Matrices)** Prove that $(AB)^t = BA$ for symmetric $n \times n$ matrices A and B.

Solution to 11a. To prove that $(AB)^t = BA$ for symmetric $n \times n$ matrices A and B, begin with the theorem $(AB)^t = B^t A^t$ and use the hypothesis $A^t = A$, $B^t = B$.

12. **(False Determinant Rules)** Give an example in dimension 2 where $\det(A + B) \neq \det(A) + \det(B)$.

Solution to 12. There are many examples in dimension 2 where $\det(A+B) \neq \det(A)+\det(B)$. The easiest to find is $A = B = I$. There are many others.

13. **(Determinant of a Symmetric Matrix)** Assume $A^{-1} = A^t$. Prove that $\det(A) = \pm 1$.

Solution to 13. Assume $A^{-1} = A^t$. We are to prove that $\det(A) = \pm 1$. The determinant rules $\det(AB) = \det(A)\det(B)$ and $\det(A) = \det(A^t)$ can be combined to show $\det(A)^2 = 1$, hence the claimed result.

13a. **(Permutation Matrices** Let P be a 3×3 matrix obtained from the identity matrix by interchanging columns. Argue from the cofactor expansion rule that $\det(P) = \pm 1$.

Solution to 13a. Let P be a 3×3 matrix obtained from the identity matrix by interchanging columns. Then P must be one of the six matrices below:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

The cofactor expansion rule can be used to evaluate each of the six determinants and verify in each case that $\det(P) = \pm 1$.

14. **(Characteristic Equation)** Prove that $\det(A - \lambda I)$ equals

$$(-\lambda)^3 + \mathrm{trace}(A)(-\lambda)^2 + (\sum_{i=1}^{3} M_{ii})(-\lambda) + \det(A)$$

where M_{ij} is the minor determinant of element ij of matrix A and trace(A) is the sum of the diagonal elements of A.

Solution to 14. The expansion of

$$\det(A - \lambda I) = \begin{vmatrix} a_{11} - \lambda & a_{12} & a_{13} \\ a_{21} & a_{22} - \lambda & a_{23} \\ a_{31} & a_{32} & a_{33} - \lambda \end{vmatrix}$$

using the cofactor expansion rule is long and tedious, but direct.

14a. **(Vandermonde Determinant)** Let

$$A = \begin{pmatrix} 1 & 1 & 1 \\ x & y & z \\ x^2 & y^2 & z^2 \end{pmatrix}.$$

Prove that $\det(A) = (y - x)(z - x)(z - y)$, by viewing the determinant as a quadratic polynomial in x having roots y and z.

Solution to 14a. Let

$$f(x) = \det \begin{pmatrix} 1 & 1 & 1 \\ x & y & z \\ x^2 & y^2 & z^2 \end{pmatrix}.$$

Then

$$f(y) = \det \begin{pmatrix} 1 & 1 & 1 \\ y & y & z \\ y^2 & y^2 & z^2 \end{pmatrix} = 0, \quad f(z) = \det \begin{pmatrix} 1 & 1 & 1 \\ z & y & z \\ z^2 & y^2 & z^2 \end{pmatrix} = 0,$$

because a determinant vanishes if two columns are the same. By the cofactor expansion rule, $f(x)$ is a quadratic polynomial in x, and since two roots are known, $f(x) = c(x - y)(x - z)$, for some c, by the factor theorem of college algebra. Its easy to check that c is the coefficient of x^2 in the cofactor expansion, and this can be evaluated directly as $(z - y)$. Left out of this solution are the required page references to relevant textbooks (for the theorems used) and display of the required calculations .

14b. **(Determinants)** Prove that the determinant of any product of upper and lower triangular matrices is the product of the diagonal entries of all the matrices involved.

Solution to 14b. The result depends upon the formula $\det(AB) = \det(A)\det(B)$, valid for $n \times n$ matrices A and B. The formula implies that the determinant of a product of matrices is the product of their individual determinants. To finish the proof, it suffices to show that the determinant of a triangular matrix is the product of its diagonal elements. This last result is done by appeal to the cofactor expansion rule.

15. **(Determinants)** Evaluate $\det(A)$:

$$A = \begin{pmatrix} a & b & 0 & 0 \\ c & d & 0 & 0 \\ 0 & 0 & a & -b \\ 0 & 0 & c & d \end{pmatrix}$$

Solution to 15. The answer:

$$\det\begin{pmatrix} a & b & 0 & 0 \\ c & d & 0 & 0 \\ 0 & 0 & a & -b \\ 0 & 0 & c & d \end{pmatrix} = (ad)^2 - (bc)^2$$

because

$$\det(A) = \det\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \det\begin{pmatrix} a & -b \\ c & d \end{pmatrix} = (ad - bc)(ad + bc)$$

16. **(Cramer's Rule)** Solve by Cramer's rule: $x_1 + x_2 + x_3 = 6$, $2x_1 - x_2 = 0$, $2x_1 + x_3 = 1$.

16a. **(Cramer's Rule)** Solve by Cramer's Rule:

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 2 & 0 & -1 & -1 \\ 0 & 0 & 3 & 6 \\ 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} 6 \\ 4 \\ 3 \\ 5 \end{pmatrix}.$$

16b. **(Cramer's Rule)** Use Cramer's rule to calculate the unknown x_3 if x_1, x_2 and x_3 satisfy the following system of linear equations:

$$\begin{array}{rcrcrcr} 2x_1 & - & x_2 & + & 2x_3 & = & 2 \\ x_1 & + & 10x_2 & - & 3x_3 & = & 5 \\ -x_1 & + & x_2 & + & 5x_3 & = & -7 \end{array}$$

Solution to 16b. By Cramer's rule, the unknown x_3 is given by $\det(A_3)/\det(A)$, where

$$A = \begin{pmatrix} 2 & -1 & 2 \\ 1 & 10 & -3 \\ -1 & 1 & 5 \end{pmatrix}, \quad A_3 = \begin{pmatrix} 2 & -1 & 2 \\ 1 & 10 & 5 \\ -1 & 1 & -7 \end{pmatrix}.$$

Therefore, $x_3 = -1$.

17. **(Inverse by Two Methods)** Show that the matrix

$$A = \begin{pmatrix} -2 & 1 & 1 \\ 0 & 1 & 1 \\ -3 & 0 & 6 \end{pmatrix}$$

is invertible and find the inverse matrix A^{-1} by two methods.

Solution to 17. The answer is

$$A^{-1} = \begin{pmatrix} -2 & 1 & 1 \\ 0 & 1 & 1 \\ -3 & 0 & 6 \end{pmatrix}^{-1} = \begin{pmatrix} -1/2 & 1/2 & 0 \\ 1/4 & 3/4 & -1/6 \\ -1/4 & 1/4 & 1/6 \end{pmatrix}.$$

The first method is the RREF method, in which $\text{rref}((A : I))$ is computed, giving $\left(I : A^{-1}\right)$. The second method is the adjoint method, which amounts to computing one 3×3 determinant and six 2×2 determinants.

Kindly show all details, by hand. The adjoint matrix (transpose of the matrix of cofactors) and the 3×3 determinant are given by

$$\text{adjoint}(A) = \begin{pmatrix} 6 & -6 & 0 \\ -3 & -9 & 2 \\ 3 & -3 & -2 \end{pmatrix}, \quad \det(A) = -12.$$

17a. **(Inverse by the Adjoint Method)** Compute by the adjoint method the inverse of the 4×4 Hilbert matrix

$$A = \begin{pmatrix} 1 & 1/2 & 1/3 & 1/4 \\ 1/2 & 1/3 & 1/4 & 1/5 \\ 1/3 & 1/4 & 1/5 & 1/6 \\ 1/4 & 1/5 & 1/6 & 1/7 \end{pmatrix}$$

Show all steps in the evolution of the solution, in particular, show explicitly the computation of $\det(A)$ and the 16 cofactors, and exhibit the final transposition of the matrix of cofactors.

Solution to 17a. The **adjoint method for the inverse** refers to the formula

$$A^{-1} = \frac{1}{\det(A)} \text{adjoint}(A),$$

where adjoint(A) is the transpose of the matrix of cofactors of A. This problem is tedious without a computer algebra system, therefore, it is in your best interest to use `maple` for many of the steps.

The determinant of A is $\det(A) = \dfrac{1}{6048000}$. Please show the computation steps for this determinant, using the cofactor expansion rule.

The adjoint matrix is given by

$$\text{adjoint}(A) = \begin{pmatrix} \frac{1}{378000} & -\frac{1}{50400} & \frac{1}{25200} & -\frac{1}{43200} \\ -\frac{1}{50400} & \frac{1}{5040} & -\frac{1}{2240} & \frac{1}{3600} \\ \frac{1}{25200} & -\frac{1}{2240} & \frac{3}{2800} & -\frac{1}{1440} \\ -\frac{1}{43200} & \frac{1}{3600} & -\frac{1}{1440} & \frac{1}{2160} \end{pmatrix}.$$

Show all 16 steps in computing this matrix. Do the first one by hand and the others by machine. For example, the matrix formed from A by deleting row 1 and column 1 produces the first cofactor C_{11} as follows:

$$M_{11} = \begin{pmatrix} 1/3 & 1/4 & 1/5 \\ 1/4 & 1/5 & 1/6 \\ 1/5 & 1/6 & 1/7 \end{pmatrix}, \quad \det(M_{11}) = \frac{1}{378000}, \quad C_{11} = (-1)^{1+1}\det(M_{11}) = \frac{1}{378000}.$$

The matrix of cofactors is $[C_{ij}]$, but adjoint(A) is not this matrix, but instead the transpose! Leaving out a step produces the wrong matrix, but for this example, the inverse is symmetric, and the classic mistake (forgetting the transpose) does not surface.

The answer for the inverse is

$$A^{-1} = \frac{1}{\det(A)} \text{adjoint}(A) = \begin{pmatrix} 16 & -120 & 240 & -140 \\ -120 & 1200 & -2700 & 1680 \\ 240 & -2700 & 6480 & -4200 \\ -140 & 1680 & -4200 & 2800 \end{pmatrix}.$$

Some `maple` hints. The command `A:=hilbert(4);` enters the matrix. And `minor(A,1,1);` produces the matrix formed by deleting row 1 and column 1. Evaluate determinants with `det(B);` where B is a square matrix. The maple command `transpose(B);` is used to form the transpose of a matrix B. The command `adjoint(A);` computes the transpose of the matrix of cofactors of A.
The purpose of the exercise is to learn how to deal with large problems. Do enough hand computation to feel comfortable. Be driven to `maple` by tedium, after you have already obtained many of the correct answers by hand.

18. **(Independence of Vectors)** Use determinants and textbook theorems to determine whether the following vectors are linearly dependent.

$$a_1 = \begin{pmatrix} -5 \\ 1 \\ 0 \end{pmatrix}, \quad a_2 = \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}, \quad a_3 = \begin{pmatrix} 6 \\ 1 \\ 0 \end{pmatrix}.$$

Solution to 18. According to Cramer's Rule, the vectors are linearly independent if and only if the matrix A whose columns are a_1, a_2, a_3 has rank 3. This can be tested effectively with the RREF.

$$A = \begin{pmatrix} -5 & 1 & 6 \\ 1 & -1 & 1 \\ 0 & -1 & 0 \end{pmatrix}, \quad \mathrm{rref}(A) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

The rank is 3, therefore the vectors are linearly independent.

18a. **(Divisibility)** Let the 3×3 matrix A be formed by taking as its rows the 9 digits of three 3-digit integers, e.g., for 228, 266, 323 the matrix is

$$A = \begin{pmatrix} 2 & 2 & 8 \\ 2 & 6 & 6 \\ 3 & 2 & 3 \end{pmatrix}.$$

Prove using Cramer's rule: if an integer m divides each number, then m divides $\det(A)$ (e.g., 19 divides 228, 266 and 323 implies 19 divides $\det(A)$ in the illustration).

Solution to 18a. As an illustration, let

$$A = \begin{pmatrix} 2 & 2 & 8 \\ 2 & 6 & 6 \\ 3 & 2 & 3 \end{pmatrix}, \quad b = \begin{pmatrix} 228 \\ 266 \\ 323 \end{pmatrix}, \quad X = \begin{pmatrix} 100 \\ 10 \\ 1 \end{pmatrix}.$$

Then the system of equations $AX = b$ is satisfied. According to Cramer's rule, each of the integer entries of X is a quotient $\det(C)/\det(A)$ for some matrix C. In particular, the third component of X equal to one implies that $\det(A) = \det(C)$, where C is the matrix A with the last column replaced by b. Write the last column of C in factored form (19 factors out of each entry) and see what it says.

Maple notes on problem 11-18a. The method for computing an inverse matrix suggested in most linear algebra texts involves augmentation of the original matrix A with the identity matrix I of the same size to create a new matrix C. This matrix C is subjected to $RREF$ to determine the inverse A^{-1}.
A second method is possible, which is based upon Cramer's Rule. The formula

$$A^{-1} = (1/\det(A))\mathrm{adjoint}(A)$$

is reproduced by these hand calculations:

(a) Compute the matrix M of **minors** of A, i.e., M_{ij} is the minor determinant of element a_{ij} in matrix A.

(b) Introduce signs into the elements of M by the **Checkerboard Rule**: element M_{ij} gets a negative sign if $i + j$ is odd. The new matrix is called C; it is the matrix of **cofactors** of A.

(c) Transpose the matrix C to obtain the **adjoint** matrix of A, called D. `Maple` can produce this matrix directly from A by the command `D:=adjoint(A)`.

(d) Compute the **determinant** $\det(A)$ and divide it into each element of the adjoint D; this is the inverse A^{-1}.

`Maple` can compute the determinant with command `det(A)` and the inverse in one step with the command `inverse(A)`.
The classical hand computation of Cramer's Rule should be learned by everyone because it appears often in scientific literature. To this end we consider the following `maple` example:

```
with(linalg):
a:=matrix([[1,0,-1],[-1,1,0],[0,0,-1]]);
b:=vector([3,-5,2]);
#The column b replaces columns 1, 2, 3 of matrix a:
a1:=augment(b,col(a,2),col(a,3)); # Replace col 1 by b
a2:=augment(col(a,1),b,col(a,3)); # Replace col 2 by b
a3:=augment(col(a,1),col(a,2),b); # Replace col 3 by b
#The answers x, y, z are quotients of determinants:
x:=det(a1)/det(a); y:=det(a2)/det(a); z:=det(a3)/det(a);

      [  1  0  -1 ]
 a := [ -1  1   0 ]    b := [ 3, -5, 2 ]
      [  0  0  -1 ]

      [  3  0  -1 ]          [  1   3  -1 ]          [  1  0   3 ]
a1 := [ -5  1   0 ]    a2 := [ -1  -5   0 ]    a3 := [ -1  1  -5 ]
      [  2  0  -1 ]          [  0   2  -1 ]          [  0  0   2 ]
# Solve aX=b for X=vector([x,y,z]):
      x := 1 y := -4 z := -2
```

The **rank** of a matrix A is the number of nonzero rows in the Reduced Row Echelon form of A. The **nullity** of A is the number of variables minus the rank, which in the case of a square matrix, equals the number of zero rows in the RREF. It is a common and fatal error to compute the nullity as the number of zero rows in the RREF (it only applies when the matrix happens to be square)!
There is a maple command `rank` which applies to compute the rank of a matrix A: `rank(A)`. There is presently no command for the nullity, because it depends upon the number of variables, and only you can know if the given A is augmented or not. Be warned that application of the `rank` command to an augmented matrix can fail to give the correct answer: the augmented column may produce an inconsistent RREF and hence an incorrect count of nonzero rows!
There are various `maple` commands available for computing the **rank** and **kernel** (or `nullspace`) of a matrix A. They are: `rref`, `gausselim`, `rank`, `kernel`. The first two produce forms from which the rank can be deduced. The command `rank(A)` gives this number directly. A basis for the solutions of $Ax = 0$ can be found by the command `kernel(A)`. A common error with the latter command is to apply it to an augmented matrix, which casts the problem $Ax = 0$ into the wrong space dimension.
`Maple` can be used to compute a basis for the **row space** of a matrix A. The command is `rowspace(A)`. The **column space** of a matrix A has a basis which can be obtained by the `maple` command `colspace(A)`. The **nullspace** of matrix A is the set of all solutions x of the equation $Ax = 0$. A basis for the nullspace is obtained by the `maple` command `nullspace(A)` or `kernel(A)` (the terms *nullspace* and *kernel* are equivalent). Here is an example:

```
with(linalg):
A := matrix(3,2,[2,0,3,4,0,5]);
rowspace(A);
colspace(A);
kernel(A);
kernel(transpose(A));
A1:=rref(A);
A2:=rref(transpose(A));
#
#            [ 2  0 ]
#        A := [ 3  4 ]
#            [ 0  5 ]
#
#       {[ 1, 0 ], [ 0, 1 ]}
#
#       {[1, 0, -15/8 ], [ 0, 1, 5/4 ]}
#
#       {}
#
#       {[ 15/8, -5/4, 1 ]}
#
#          [ 1  0 ]            [ 1  0  -15/8 ]
#    A1 := [ 0  1 ]      A2 := [               ]
#          [ 0  0 ]            [ 0  1    5/4  ]
```

As is apparent from this example, the commands `colspace` and `rowspace` can be replaced by extraction of nonzero rows from the reduced row echelon forms for A and A^t.

Eigenvalues and Eigenvectors. Find the eigenvalues, eigenvectors, geometric multiplicity and algebraic multiplicity.

19. (2 **Eigenvalues and** 3 **Eigenvectors**)

$$A = \begin{pmatrix} 5 & 4 & 2 \\ 4 & 5 & 2 \\ 2 & 2 & 2 \end{pmatrix}.$$

Solution to 19. To find the eigenvalues of the matrix

$$A = \begin{pmatrix} 5 & 4 & 2 \\ 4 & 5 & 2 \\ 2 & 2 & 2 \end{pmatrix}$$

it is required to solve for the roots of the characteristic equation

$$0 = \det(A - \lambda I) = 10 - 21\,\lambda + 12\,\lambda^2 - \lambda^3 = (10 - \lambda)\,(\lambda - 1)^2$$

The eigenvalues are therefore 10, 1 and 1. The eigenvector for $\lambda = 10$ is found by solving $AX = 10X$ or equivalently $(A - 10I)X = 0$, which is a null space problem. The RREF of $A - (10)I$ is found (with some effort) and we obtain eigenvalue, eigenvector pair

$$\lambda = 10, \quad X = \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}.$$

In a similar way, we find the RREF of $A - (1)I$ has a basis of two elements, giving the eigenvalue, eigenvector pairs

$$\lambda = 1, \quad X = \begin{pmatrix} 0 \\ 1 \\ -2 \end{pmatrix}, \qquad \lambda = 1, \quad X = \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix}.$$

Left out of these notes is the tedious task of finding RREF's and standard forms of the solution for two null space problems (one problem for each eigenvalue). In a student solution, these details must *not* be left out!

19a. (1 **Eigenvalue and** 1 **Eigenvector**)

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -3 & 3 \end{pmatrix}.$$

Solution to 19a. To find the eigenvalues of the matrix

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -3 & 3 \end{pmatrix}$$

requires solving for the roots of the characteristic equation

$$0 = \det(A - \lambda I) = 3\,\lambda^2 - \lambda^3 - 3\,\lambda + 1 = (1 - \lambda)^3.$$

There is only one eigenvalue and therefore only one null space problem to solve, namely $AX = X$ or $(A-I)X = 0$. The RREF of $A - I$ is found to have rank 2, nullity 1, so there is only one eigenvalue, eigenvector pair:

$$\lambda = 1, \quad X = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

20. (1 **Eigenvalue and** 1 **Eigenvector**)

$$A = \begin{pmatrix} -3 & -7 & -5 \\ 2 & 4 & 3 \\ 1 & 2 & 2 \end{pmatrix}.$$

Solution to 20. To find the eigenvalues of the matrix

$$A = \begin{pmatrix} -3 & -7 & -5 \\ 2 & 4 & 3 \\ 1 & 2 & 2 \end{pmatrix}$$

requires solving for the roots of the characteristic equation

$$0 = \det(A - \lambda I) = (1 - \lambda)^3.$$

There is only one eigenvalue, eigenvector pair:

$$\lambda = 1, \quad X = \begin{pmatrix} -3 \\ 1 \\ 1 \end{pmatrix}.$$

20a. (3 **Eigenvalues and** 3 **Eigenvectors**)

$$A = \begin{pmatrix} 5 & 0 & 2 \\ 0 & 7 & -2 \\ 2 & -2 & 6 \end{pmatrix}.$$

Solution to 20a. To find the eigenvalues of the matrix

$$A = \begin{pmatrix} 5 & 0 & 2 \\ 0 & 7 & -2 \\ 2 & -2 & 6 \end{pmatrix}.$$

requires solving for the roots of the characteristic equation

$$0 = \det(A - \lambda I) = (\lambda - 3)\,(6 - \lambda)\,(\lambda - 9)\,.$$

The eigenpairs are

$$\lambda = 3, \quad X = \begin{pmatrix} -2 \\ 1 \\ 2 \end{pmatrix}, \qquad \lambda = 6, \quad X = \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}, \qquad \lambda = 9, \quad X = \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix}.$$

Maple notes on problems 19–20a. The notions of **algebraic multiplicity** and **geometric multiplicity** are integer counts taken from certain calculations. Both require that the **characteristic equation** be solved:

$$\det(A - \lambda I) = 0.$$

The *algebraic multiplicity* of λ is the number of times root λ is repeated. For example, in the equation

$$(\lambda - 1)^3(\lambda + 3)^2(\lambda^2 + 16) = 0$$

the roots $1, -3, 4i, -4i$ are repeated 3, 2, 1, 1 times respectively, hence their algebraic multiplicities are 3, 2, 1, 1.

The *geometric multiplicity* of a root λ of the characteristic equation is the number of independent eigenvectors for λ, that is, the number of independent solutions x to the equation $(A - \lambda I)x = 0$. This number can be found *without* computing eigenvectors. Precisely, the **geometric multiplicity** of root λ is the number of arbitrary variables in the general solution. This number is exactly the **nullity** of the matrix $A - \lambda I$, which is the number of variables minus the **rank**. If the eigenvectors are not needed, then it suffices to compute `rank(A-λ I)`. If the eigenvectors are actually needed, then `maple` determines the count as the number of basis vectors in the calculation `kernel(D)` where $D = A - \lambda I$. A third way to obtain the count is to apply the `maple` command `eigenvects(A)`, which contains the desired count in an encrypted syntax (along with additional information). Here is an example which shows how to compute the eigenvalues and eigenvectors and incidentally calculate the algebraic and geometric multiplicities:

```
with(linalg):
A:=matrix([[1,3,-2,0],[2,6,-5,-2],[0,0,5,10],[2,6,0,8]]);
J:=diag(1,1,1,1);
#
#       [ 1  3  -2   0 ]           [ 1  0  0  0 ]
#       [ 2  6  -5  -2 ]           [ 0  1  0  0 ]
#  A := [              ]       J:= [            ]
#       [ 0  0   5  10 ]           [ 0  0  1  0 ]
#       [ 2  6   0   8 ]           [ 0  0  0  1 ]
#
u:=[eigenvals(A)];              # Make an eigenvalue list
#                               # Brackets are not a mistake!
#                                         1/2           1/2
#                    u := [0, 0, 10 + I 43   , 10 - I 43   ]
#
# Algebraic multiplicities are 2,1,1 because 0 is repeated.
#
# Now solve the kernel problems for all eigenvalues u[1]..u[4].
```

```
#
v1:=kernel(evalm(A-u[1]*J));
v2:=kernel(evalm(A-u[2]*J));        # Duplicate computation!
v3:=kernel(evalm(A-u[3]*J));
v4:=kernel(evalm(A-u[4]*J));
#                  v1:={[ -3, 1, 0, 0 ], [ -4, 0, -2, 1 ]}
#                  v2:={[ -3, 1, 0, 0 ], [ -4, 0, -2, 1 ]}
evalf(map(evalc,v3[1]),3);
#            [  - .150 - .328 I,  - .500 - .656 I, 1., .500 - .656 I ]
evalf(map(evalc,v4[1]),3);
#            [  - .150 - .328 I,  - .500 - .656 I, 1., .500 - .656 I ]
#
#       Geometric multiplicities:       ev=0                  mult=2
#                                       ev=10.00 + 6.56 I     mult=1
#                                       ev=10.00 - 6.56 I     mult=1
```

It is possible to compute the eigenvalues of a matrix A numerically by the maple command `Eigenvals`, rather than symbolically, as is done by `eigenvals(A)`. This is recommended for those cases when the `maple` output from `eigenvals(A)` is too complicated to read. The answers usually require interpretation, as in the following, where one eigenvalue of 0 is computed as -0.874×10^{-9}:

```
with(linalg):
A:=matrix([[1,3,-2,0],[2,6,-5,-2],[0,0,5,10],[2,6,0,8]]);
evalf(Eigenvals(A),3);          # Numeric eigenvalues
                                # are not used for computations!
#                           -9
#                [ -.874*10   , 10.00 + 6.56 I, 10.00 - 6.56 I, 0 ]
#
```

21. **(Diagonalization)** Test for diagonalizability and find the diagonal form.

$$A = \begin{pmatrix} 3 & -1 & -1 \\ 1 & 1 & -1 \\ 1 & -1 & 1 \end{pmatrix}.$$

Solution to 21. A practical test for diagonalizability of a square matrix A is as follows:

(a) Compute all eigenvalues of A. If they are distinct, then A is diagonalizable.
(b) If test (a) fails, then compute the eigenvectors of A. If the number of independent eigenvectors equals the dimension of A then A is diagonalizable.

If either case (a) or (b) holds, then A is diagonalizable and its diagonal form is the diagonal matrix of eigenvalues. The `maple` command `Eigenvals(A)` (Cap E, not lowercase e) is not very useful for deciding case (a) because the numerical values may be distinct but the actual values identical — see the example above. The command `eigenvects(A)` can be used to decide case (b). By standard theory, case (b) holds whenever case (a) holds, so you might deduce that case (a) can be eliminated. However, computational complexity often dictates that (a) be checked first.
Diagonalizability for

$$A = \begin{pmatrix} 3 & -1 & -1 \\ 1 & 1 & -1 \\ 1 & -1 & 1 \end{pmatrix}$$

is tested by first computing the eigenvalues. They are 1, 2, 2, not distinct, so we have to check the geometric multiplicity of $\lambda = 2$. If it's 2, then the total geometric multiplicity is 3 and A is diagonalizable with diagonal form

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

Otherwise, its not diagonalizable. The geometric multiplicity question is equivalent to finding the nullity of $A - 2I$. So we find the rank of $A - 2I$ and subtract from 3 (the number of variables). A quick check gives rank 1, so the nullity is 2 and A is diagonalizable.

21a. (Diagonalization) Test for diagonalizability and find the diagonal form.

$$A = \begin{pmatrix} -2 & -2 & 0 & 0 \\ -5 & 1 & 0 & 0 \\ 0 & 0 & 2 & -1 \\ 0 & 0 & 5 & -2 \end{pmatrix}.$$

Solution to 21a. Diagonalizability for

$$A = \begin{pmatrix} -2 & -2 & 0 & 0 \\ -5 & 1 & 0 & 0 \\ 0 & 0 & 2 & -1 \\ 0 & 0 & 5 & -2 \end{pmatrix}.$$

is tested by first computing the eigenvalues. They are 3, -4, i and $-i$ ($i = \sqrt{-1}$). So the eigenvalues are distinct and A is diagonalizable with diagonal form

$$\begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & -4 & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & 0 & 0 & -i \end{pmatrix}.$$

22. (Orthogonal Matrices) Find an orthogonal matrix Q such that $Q^t AQ$ is diagonal:

$$A = \begin{pmatrix} 1 & -1 & 0 & 0 \\ -1 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}.$$

Solution to 22. A square matrix Q is **orthogonal** if each column has length 1 and the columns of Q are pairwise orthogonal, that is, $X \cdot Y = 0$ for any pair of columns X and Y of Q.
The matrix P of eigenvectors of a symmetric matrix A already satisfies $P^{-1}AP = D$, where D is the diagonal matrix of eigenvalues. An orthogonal matrix Q is constructed from P by changing its columns to unit vectors, accomplished by dividing each column by its length. Then Q is orthogonal, $Q^{-1} = Q^t$, and the equation $Q^t AQ = D$ holds.
The eigenvalues and eigenvectors of the matrix

$$A = \begin{pmatrix} 1 & -1 & 0 & 0 \\ -1 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}$$

are found to be

$$\lambda = 2 + \sqrt{2}, \quad X = \begin{pmatrix} 1 - \sqrt{2} \\ 1 \\ 0 \\ 0 \end{pmatrix}, \qquad \lambda = 2 - \sqrt{2}, \quad X = \begin{pmatrix} 1 + \sqrt{2} \\ 1 \\ 0 \\ 0 \end{pmatrix},$$

$$\lambda = 0, \quad X = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \qquad \lambda = 2, \quad X = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

Let $a = \sqrt{4 - 2\sqrt{2}}$, $b = \sqrt{4 + 2\sqrt{2}}$ be the norms of the first two eigenvectors. The matrix Q of *normalized eigenvectors* is given by

$$Q = \begin{pmatrix} (1-\sqrt{2})/a & (1+\sqrt{2})/b & 0 & 0 \\ 1/a & 1/b & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

It is possible to show directly that the columns of Q have unit length and are pairwise orthogonal. Therefore, Q is orthogonal, which means $Q^{-1} = Q^t$. Finally, Q^tAQ is the diagonal matrix of eigenvalues:

$$\begin{pmatrix} 2+\sqrt{2} & 0 & 0 & 0 \\ 0 & 2-\sqrt{2} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}.$$

Maple notes on 19–22. A symmetric matrix A with distinct eigenvalues can be transformed to diagonal form $D = \text{diag}(\lambda_1, \ldots, \lambda_n)$ by the matrix Q of its normalized eigenvectors: $Q^{-1}AQ = D$. The algorithm for finding Q is as follows:

(a) Let `u:=[eigenvals(A)]`. The values should be distinct.

(b) Evaluate

```
v:=kernel(evalm(A-u[i]*J))
```

for $i = 1, \ldots, n$. Save the n columns

```
evalm(v/sqrt(dotprod(v,v)))
```

in a matrix Q using initially `Q:=v` and then `Q:=augment(Q,v)` to add columns one at a time until n columns are filled.

There is the possibility of using `eigenvects(A)` to produce the values `v` of step (b) above. Sometimes `norm(v,2)` produces unevaluated absolute values — use `sqrt(dotprod(v,v))` instead of `norm(v,2)`.
It is remarked that the above algorithm applies only to symmetric matrices. Application to nonsymmetric matrices is considered a logical error (the calculation may produce no error message but the answer is likely incorrect).

```
with(linalg):
A:=matrix([[16,-3/2],[-3/2,4]]);
v:=[eigenvals(A)]; n:=coldim(A): J:=diag(seq(1,i=1..n)):
for i from 1 to n do
  w:=kernel(evalm(A-v[i]*J));
  y:=evalm(w[1]/sqrt(dotprod(w[1],w[1]))):
  y:=evalf(map(evalc,y),3):
  if i=1 then Q:=matrix(n,1,y): else
     Q:=augment(Q,matrix(n,1,y)): fi:
od:
Q = evalf(eval(Q),3);
#
# Applies only to symmetric matrices!
#
```

```
#        [  16   -3/2 ]             [  .990  .122 ]
#  A := [            ]         Q = [             ]
#        [ -3/2    4  ]             [ -.119  .991 ]
#
#                                        1/2             1/2
#                     v := [10 + 3/2 17     , 10 - 3/2 17    ]
#
```

Quadratic Forms. Write the given quadratic as as $AX \cdot X$ for some symmetric matrix A.

23. **(Quadratic Form in R^4)** $x_1^2 - x_2^2 + x_1x_3 - x_2x_4 + x_3^2 + x_4^2$

Solution to 23. The quadratic $x_1^2 - x_2^2 + x_1x_3 - x_2x_4 + x_3^2 + x_4^2$ transforms to $AX \cdot X$ where

$$X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}, \quad A = \begin{pmatrix} 1 & 0 & 1/2 & 0 \\ 0 & -1 & 0 & -1/2 \\ 1/2 & 0 & 1 & 1 \\ 0 & -1/2 & 0 & 1 \end{pmatrix}.$$

Example. Write the quadratic form

$$x_1^2 - x_2^2 + x_1x_3 - 3x_2x_4 + 4x_3^2 + x_4^2 = 10$$

in the matrix form $X^tAX = 10$ for some symmetric matrix A.
Define

$$X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}, \quad A = \begin{pmatrix} 1 & 0 & 1/2 & 0 \\ 0 & -1 & 0 & -3/2 \\ 1/2 & 0 & 4 & 0 \\ 0 & -3/2 & 0 & 1 \end{pmatrix}.$$

The trick in defining A is is to assign diagonal entries in A to corresponding square terms in the quadratic form, but to cross terms like $-3x_2x_4$ assign **two** off–diagonal entries (e.g., assign $-3/2$ to symmetric entries a_{24} and a_{42} of A).

23a. **(Quadratic Form in R^3)** $-x^2 + xy + y^2 - 4xz + 4yz + z^2$

Solution to 23a. The quadratic $-x^2 + xy + y^2 - 4xz + 4yz + z^2$ transforms to $AX \cdot X$ where

$$X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \quad A = \begin{pmatrix} -1 & 1/2 & -2 \\ 1/2 & 1 & 2 \\ -2 & 2 & 1 \end{pmatrix}.$$

Maple notes on 23–23a. To calculate the orthogonal matrix R (the **rotation** matrix) that transforms a quadratic form $x^tAx = c$ into its **normal form** $X^tDX = c$ (D is a diagonal matrix), the algorithm for symmetric matrices is followed to create an orthogonal matrix Q of eigenvectors of A such that $D = Q^{-1}AQ$ is the diagonal matrix of eigenvalues of A. Then R is the matrix Q of that algorithm and the normal form results by taking the change of variables $X = Qx$.

Example. Find the rotation matrix Q and the standard form for the quadratic $16x^2 - 3xy + 4y^2 = 10$, using `maple`.
Solution:

```
with(linalg):
k:=10:eq:= 16*x^2 - 3*x*y + 4*y^2=k;         # The problem
A:=matrix([[16,-3/2],[-3/2,4]]);
v:=[eigenvals(A)]; n:=coldim(A): J:=array(identity,1..n,1..n):
for i from 1 to n do
  w:=kernel(evalm(A-v[i]*J));
  y:=evalm(w[1]/sqrt(dotprod(w[1],w[1]))):
  y:=evalf(map(evalc,y),3):
  if i=1 then Q:=matrix(n,1,y): else
     Q:=augment(Q,matrix(n,1,y)): fi:
od:
X:=matrix(n,1,[X1,X2]):                  # New variables of standard form
eval(X)=evalf(eval(Q),3)*matrix(n,1,[x1,x2]);   # Rotation formulas
DD:=diag(seq(v[j],j=1..n));
FORM:=evalm(transpose(X) &* DD &* X);   # A 1 by 1 matrix
collect(FORM[1,1],[X1,X2])=k;           # Write in standard form
#                                 2              2
#                         eq := 16 x  - 3 x y + 4 y  = 10
#
#                                  [  16   -3/2 ]
#                             A := [            ]
#                                  [ -3/2    4  ]
#
#                                     1/2             1/2
#                      v := [10 + 3/2 17   , 10 - 3/2 17   ]
#
#                  [ X1 ]    [  .990    .119  ] [ x1 ]
#                  [    ]  = [                ] [    ]
#                  [ X2 ]    [ -.119    .990  ] [ x2 ]
#
#                           [           1/2                 ]
#                           [ 10 + 3/2 17           0       ]
#                      D := [                               ]
#                           [                           1/2 ]
#                           [        0        10 - 3/2 17   ]
#
#                              1/2    2                1/2    2
#                 (10 + 3/2 17   ) X1  + (10 - 3/2 17   ) X2  = 10
#
```

Standard Form of a Quadratic. Write the quadratic in standard form $\lambda_1 X^2 + \lambda_2 Y^2 = c$ and display both the rotation matrix R and the standard form.

24. **(Quadratic Forms)** $3x^2 - 2xy = 5$

Solution to 24. The normal form of $3x^2 - 2xy = 5$ is

$$\left(3/2 + \frac{\sqrt{13}}{2}\right) X1^2 + \left(3/2 - \frac{\sqrt{13}}{2}\right) X2^2 = 5, \quad \begin{pmatrix} X1 \\ X2 \end{pmatrix} = \begin{pmatrix} 0.96 & 0.29 \\ -0.29 & 0.96 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

24a. **(Quadratic Forms)** $x^2 - 3xy + 4y^2 = 1$

Solution to 24a. The normal form of $x^2 - 3xy + 4y^2 = 1$ is

$$\left(5/2 + \frac{3\sqrt{2}}{2}\right) X1^2 + \left(5/2 - \frac{3\sqrt{2}}{2}\right) X2^2 = 1, \quad \begin{pmatrix} X1 \\ X2 \end{pmatrix} = \begin{pmatrix} -0.380 & 0.923 \\ 0.926 & 0.383 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

24b. (Quadratic Forms) $2x^2 + xy + y^2 = 4$

Solution to 24b. The normal form of $2x^2 + xy + y^2 = 4$ is

$$\left(3/2 + \frac{\sqrt{2}}{2}\right) X1^2 + \left(3/2 - \frac{\sqrt{2}}{2}\right) X2^2 = 4, \quad \begin{pmatrix} X1 \\ X2 \end{pmatrix} = \begin{pmatrix} 0.923 & -0.380 \\ 0.383 & 0.926 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

Differential Equations. The general solution of a matrix differential equation $x'(t) = Ax(t)$ can be written as $x(t) = \sum_{k=1}^{n} c_k X_k e^{\lambda_k t}$ where c_1 to c_n are arbitrary constants, λ_1 to λ_n are the distinct eigenvalues of A with corresponding eigenvectors X_1 to X_n. This solution works only when A has n distinct eigenvalues.

25. (Differential Equations) Find the general solution of the system of differential equations $x'(t) = Ax(t)$ where

$$A = \begin{pmatrix} 2 & -4 & 4 \\ 0 & -2 & 1 \\ 0 & 0 & -1 \end{pmatrix}.$$

Solution to 25. To solve differential equations $x' = Ax$ where A is a square matrix we apply the standard theorem that says, for dimension 3,

$$x = c_1 v_1 \exp(\lambda_1 t) + c_2 v_2 \exp(\lambda_2 t) + c_3 v_3 \exp(\lambda_3 t)$$

where λ_1, λ_2, λ_3 are the distinct eigenvalues of the matrix A with corresponding eigenvectors v_1, v_2, v_3. The symbols c_1, c_2, c_3 represent arbitrary constants in the general solution.
The above method is applicable only in the case where A has distinct eigenvalues. Methods exist to solve the problem for a general matrix A, however, the theory is beyond the scope of the linear algebra already developed.
The general solution of the system of differential equations $x'(t) = Ax(t)$ where

$$A = \begin{pmatrix} 2 & -4 & 4 \\ 0 & -2 & 1 \\ 0 & 0 & -1 \end{pmatrix}$$

is given by

$$x(t) = c_1 \exp(2t) \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + c_2 \exp(-2t) \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + c_3 \exp(-t) \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

25a. (Differential Equations) Solve for the vector solution $x(t)$ in $x'(t) = Ax(t)$, given

$$A = \begin{pmatrix} 4 & 0 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{pmatrix}.$$

Solution to 25a. The vector solution $x(t)$ in $x'(t) = Ax(t)$, given

$$A = \begin{pmatrix} 4 & 0 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{pmatrix},$$

is given by

$$x(t) = c_1 \exp(2t) \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + c_2 \exp(4t) \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + c_3 \exp(3t) \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}$$

25b. **(Vector Space Basis)** Find a basis for the solution space of the linear differential equation $y''' - 6y'' + 11y' - 6y = 0$.

Solution to 25b. A basis for the solution space of a homogeneous linear differential equation can be extracted from the **General Solution** of the equation. The basis is obtained formally from the general solution by identifying the functions multiplying the arbitrary constants in the general solution.

From the general solution

$$y(x) = C_1 \exp(x) + C_2 \exp(-3x) + C_3 \exp(-2x)$$

we can infer that $\exp(x)$, $\exp(-3x)$, $\exp(-2x)$ is a basis for the solution space of the differential equation $y''' + 4y'' + y' - 6y = 0$.

A basis for the solution space of the linear differential equation $y''' - 6y'' + 11y' - 6y = 0$ is

$$e^x, \quad e^{2x}, \quad e^{3x}.$$

Maple Notes on Differential Equations. The general solution of a differential equation can be found by generalizing this example:

```
u:=y(x): u1:=diff(y(x),x):
u2:=diff(y(x),x,x): u3:=diff(y(x),x,x,x):
de:=u3+4*u2+u1-6*u = 0:
dsolve(de,y(x));
#             y(x) = _C1 exp(x) + _C2 exp(- 3 x) + _C3 exp(- 2 x)
```