# 5.7 Forced Mechanical Vibrations

The study of vibrating mechanical systems continues. The main example is a system consisting of an externally forced mass on a spring with dampener. Both undamped and damped systems are studied. A number of physical examples are given, which include the following: clothes dryer, cafe door, pet door, bicycle trailer.

## Forced undamped motion

The equation for study is a forced spring–mass system

$$mx''(t) + kx(t) = f(t).$$

The model originates by equating the Newton's second law force $mx''(t)$ to the sum of the Hooke's force $-kx(t)$ and the external force $f(t)$. The physical model is a laboratory box containing an undamped spring–mass system, transported on a truck as in Figure 11, with external force $f(t) = F_0 \cos \omega t$ induced by the speed bumps.

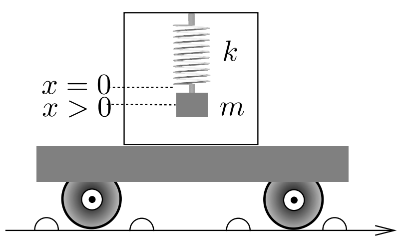


**Figure 11. An undamped spring-mass system in a box is transported on a truck. Speed bumps on the shoulder of the road induce periodic vertical oscillations to the box.**

The forced equation takes the form

$$x''(t) + \omega_0^2 \, x(t) = \frac{F_0}{m} \cos \omega t, \quad \omega_0 = \sqrt{k/m}.$$

The **natural frequency** $\omega_0$ corresponds to free oscillation of the mass, that is, the number of full periods of oscillation per second for the spring–mass system when no external force is present. The **external frequency** $\omega$ is the number of full periods of oscillation per second of the external force $f(t) = F_0 \cos \omega t$. In the case of Figure 11, this is the vertical force applied to the box containing the spring–mass system, due to the speed bumps. The general solution $x(t)$ always presents itself in two pieces, as the sum of the homogeneous solution $x_h$ and a particular solution $x_p$. For $\omega \neq \omega_0$, the general solution is

$$\begin{aligned} x(t) &= x_h(t) + x_p(t), \\ x_h(t) &= c_1 \cos \omega_0 t + c_2 \sin \omega_0 t, \quad c_1,\, c_2 \text{ constants}, \\ x_p(t) &= F_1 \cos \omega t, \quad F_1 = \frac{F_0/m}{\omega_0^2 - \omega^2}. \end{aligned} \tag{1}$$

A general statement can be made about the solution decomposition:

> The solution is a sum of two harmonic oscillations, one of natural frequency $\omega_0$ due to the spring and the other of natural frequency $\omega$ due to the external force $F_0 \cos \omega t$.

**Rapidly and slowly varying functions**. The superposition $x(t)$ will show the phenomenon of **beats** for certain choices of $\omega_0$, $\omega$, $x(0)$ and $x'(0)$. For example, consider $x(t) = \cos \omega_0 t - \cos \omega t$. Use the trigonometric identity $2 \sin a \sin b = \cos(a-b) - \cos(a+b)$ to write $x(t) = A(t) \sin \frac{1}{2}(\omega_0 + \omega)t$ where $A(t) = 2 \sin \frac{1}{2}(\omega_0 - \omega)t$. If $\omega \approx \omega_0$, then $A(t)$ has natural frequency $\alpha = \frac{1}{2}(\omega_0 - \omega)$ near zero. The natural frequency $\beta = \frac{1}{2}(\omega_0 + \omega)$ can be relatively large and therefore $x(t)$ is a product of a **slowly varying** amplitude $A(t) = 2 \sin \alpha t$ and a **rapidly varying** oscillation $\sin \beta t$.

The physical phenomenon of **beats** refers to the periodic cancellation of sound at a slow frequency. An illustration of the graphical meaning of *beats* appears in Figure 12.

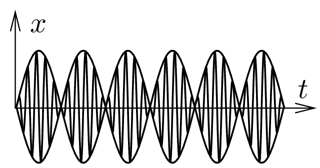


**Figure 12. The phenomenon of beats. Shown is a rapidly–varying periodic oscillation $x(t) = 2 \sin 4t \sin 40t$ and the two slowly–varying envelope curves $x_1(t) = 2 \sin 4t$, $x_2(t) = -2 \sin 4t$.**

An example is striking simultaneously two identical tuning forks, the first slightly out of tune with the second. A **destructive interference** occurs during a very brief interval, so our impression is that the sound periodically stops, only briefly, and then starts again with a *beat*, a section that is instantaneously loud again. The origin of this impression can be seen from the formula $x(t) = A(t) \sin \beta t$ where $A(t) = 2 \sin \alpha t$. There is no sound when $x(t) \approx 0$: this is when destructive interference occurs. When $\alpha$ is small compared to $\beta$, there are long intervals between the zeros of $A(t)$, at which destructive interference occurs and $x(t) \approx 0$. Otherwise, the amplitude of the sound is the average value of $A(t)$, which is 1. The sound stops at a zero of $A(t)$ and then it is rapidly loud again, causing the beat.

**Rotating drum on a cart.** Figure 13 shows a model for a rotating machine, like a front–loading clothes dryer.

For modelling purposes, the rotating drum with load is replaced by an idealized model: a mass $\mathcal{M}$ on a string of radius $R$ rotating with angular speed $\omega$. The center of rotation is located along the center–line of the cart. The total mass $m$ of the cart includes the rotating mass $\mathcal{M}$, which we imagine to be an off–center lump of wet laundry inside the dryer drum. Vibrations cause the cart to skid left or right.

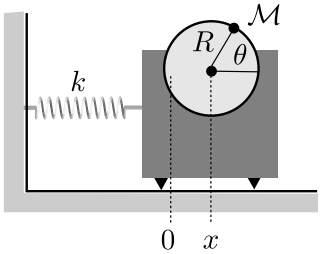


**Figure 13. A rotating vertical drum installed on a cart with skids. A spring restores the cart to its equilibrium position $x = 0$.**

A spring of Hooke's constant $k$ restores the cart to its equilibrium position $x = 0$. The cart has position $x > 0$ corresponding to skidding distance $x$ to the right of the equilibrium position, due to the off-center load. Similarly, $x < 0$ means the cart skidded distance $|x|$ to the left.

**Modelling**. Friction ignored, Newton's second law gives force $F = m\overline{x}''(t)$, where $\overline{x}$ locates the cart's center of mass. Hooke's law gives force $F = -kx(t)$. The centroid $\overline{x}$ can be expanded in terms of $x(t)$ by using calculus moment of inertia formulas. Let $m_1 = m - \mathcal{M}$ be the cart mass, $m_2 = \mathcal{M}$ the drum mass, $x_1 = x(t)$ the moment arm for $m_1$ and $x_2 = x(t) + R\cos\theta$ the moment arm for $m_2$. Then $\theta = \omega t$ in Figure 13 gives

$$
\begin{aligned}
\overline{x}(t) &= \frac{m_1x_1 + m_2x_2}{m_1 + m_2} \\
&= \frac{(m - \mathcal{M})x(t) + \mathcal{M}(x(t) + R\cos\theta)}{m} \qquad (2)\\
&= x(t) + \frac{R\mathcal{M}}{m}\cos\omega t.
\end{aligned}
$$

Force competition $m\overline{x}'' = -kx$ and derivative expansion results in the forced harmonic oscillator

$$
mx''(t) + kx(t) = R\mathcal{M}\omega^2 \cos\omega t. \qquad (3)
$$

## Forced damped motion

Real systems do not exhibit idealized harmonic motion, because **damping** occurs. A watch balance wheel submerged in oil is a key example: frictional forces due to the viscosity of the oil will cause the wheel to stop after a short time. The same wheel submerged in air will appear to display harmonic motion, but indeed there is friction present, however small, which slows the motion.

Consider a spring–mass system consisting of a mass $m$ and a spring with Hooke's constant $k$, with an added **dashpot** or **dampener**, depicted in Figure 14 as a piston inside a cylinder attached to the mass. A useful physical model, for purposes of intuition, is a screen door with door–closer: the closer has a spring and an adjustable piston–cylinder style dampener.

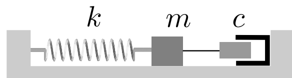


**Figure 14. A spring-mass system with dampener**

The dampener is assumed to operate in the **viscous domain**, which means that the force due to the dampener device is proportional to the speed that the mass is moving: $F = cx'(t)$. The number $c \geq 0$ is called the **damping constant**. Three forces act: (1) Newton's second law $F_1 = mx''(t)$, (2) viscous damping $F_2 = cx'(t)$ and (3) the spring restoring force $F_3 = kx(t)$. The sum of the forces $F_1 + F_2 + F_3$ acting on the system must equal the **external force** $f(t)$, which gives the equation for a **damped spring–mass system**

$$mx''(t) + cx'(t) + kx(t) = f(t). \tag{4}$$

The motion is called **damped** if $c > 0$ and **undamped** if $c = 0$. If there is no external force, $f(t) = 0$, then the motion is called **free** or **unforced** and otherwise it is called **forced**.

A useful visualization for a forced system is a vertical laboratory spring–mass system with dampener placed inside a box, which is transported down a washboard road inside an auto trunk. The function $f(t)$ is the vertical oscillation of the auto trunk. The function $x(t)$ is the motion of the mass in response to the washboard road. See Figure 15.

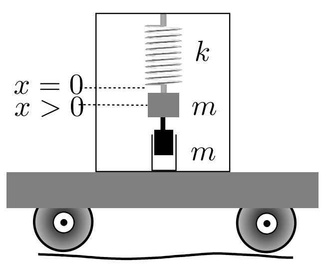


**Figure 15. A spring-mass system with dampener in a box transported in an auto trunk along a washboard road.**

**Free damped motion.** Consider the special case of no external force, $f(t) = 0$. The motion $x(t)$ satisfies the homogeneous differential equation

$$mx''(t) + cx'(t) + kx(t) = 0. \tag{5}$$

**Cafe door.** Restaurant waiters and waitresses are used to the cafe door, which blocks partially the view of onlookers, but allows rapid, collision-free trips to the kitchen – see Figure 16. The door is equipped with a spring which tries to restore the door to the equilibrium position $x = 0$, which is the plane of the door frame. There is a dampener attached, to keep the number of oscillations low.

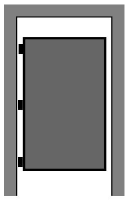

**Figure 16. A cafe door on three hinges with dampener in the lower hinge. The equilibrium position is the plane of the door frame.**

The top view of the door, Figure 17, shows how the angle $x(t)$ from equilibrium $x = 0$ is measured from different door positions.

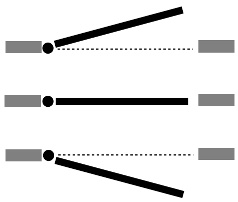


**Figure 17. Top view of a cafe door, showing the three possible door positions.**

The figure shows that, for modelling purposes, the cafe door can be reduced to a torsional pendulum with viscous damping. This results in the **cafe door** equation

$$Ix''(t) + cx'(t) + \kappa x(t) = 0. \tag{6}$$

The removal of the spring ($\kappa = 0$) causes the solution $x(t)$ to be monotonic, which is a reasonable fit to a springless cafe door.

**Pet door.** Designed for dogs and cats, the small door in Figure 18 allows animals to enter and exit the house freely. Winter drafts and summer insects are the main reasons for pet doors. Owners argue that these doors decrease damage due to clawing and beating the door to get in and out. A pet door might have a weather seal and a security lock.

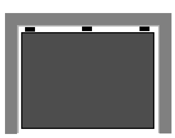

**Figure 18. A pet door. The equilibrium position is the plane of the door frame.**

The pet door swings freely from hinges along the top edge. One hinge is spring–loaded with dampener. Like the cafe door, the spring restores the door to the equilibrium position while the dampener acts to eventually stop the oscillations. However, there is one fundamental difference: if the spring–dampener system is removed, then the door continues to oscillate! The cafe door model will not describe the pet door.

For modelling purposes, the door can be compressed to a linearized swinging rod of length $L$ (the door height). The torque $I = mL^2/3$ of the door assembly becomes important, as well as the linear restoring force $kx$ of the spring and the viscous damping force $cx'$ of the dampener. All considered, a suitable model is the **pet door** equation

$$I\,x''(t) + cx'(t) + \left(k + \frac{mgL}{2}\right) x(t) = 0. \tag{7}$$

Derivation of (7) is by equating to zero the algebraic sum of the forces. Removing the dampener and spring ($c = k = 0$) gives a harmonic oscillator $x''(t) + \omega^2 x(t) = 0$ with $\omega^2 = 0.5mgL/I$, which establishes sanity for the modelling effort. Equation (7) is *formally* the cafe door equation with an added linearization term $0.5mgLx(t)$ obtained from $0.5mgL \sin x(t)$.

**Modelling.** The cafe door and the pet door have equations in the same form as a damped spring–mass system, and all three can be reduced, for suitable definitions of constants $p$ and $q$, to the simplified second order differential equation

$$x''(t) + p\,x'(t) + q\,x(t) = 0. \tag{8}$$

**Tuning a dampener.** The pet door and the cafe door have dampeners with an adjustment screw. The screw changes the damping coefficient $c$ which in turn changes the size of coefficient $p$ in (8). More damping $c$ means $p$ is larger.

There is a *critical damping effect* for a certain screw setting: if the damping is decreased more, then the door *oscillates*, whereas if the damping is increased, then the door has a *monotone non-oscillatory behavior*. The monotonic behavior can result in the door opening in one direction followed by slowly settling to exactly the door jamb position. If $p$ is too large, then it could take 10 minutes for the door to close!

The critical case corresponds to the least $p > 0$ (the smallest damping constant $c > 0$) required to close the door with this kind of monotonic behavior. The same can be said about decreasing the damping: the more $p$ is decreased, the more the door oscillations approach those of no dampener at all, which is a pure harmonic oscillation.

As viewed from the characteristic equation $r^2 + pr + q = 0$, the change is due to a change in character of the roots from real to complex. The physical response and the three cases of the constant–coefficient recipe, page 189, lead to the following terminology.

| **Classification** | **Defining properties** |
|---|---|
| Overdamped | Distinct real roots $r_1 \neq r_2$<br>Positive discriminant<br>$x = c_1 e^{r_1 t} + c_2 r^{r_2 t}$<br>$=$ exponential $\times$ monotonic function |
| Critically damped | Double real root $r_1 = r_2$<br>Zero discriminant<br>$x = c_1 e^{r_1 t} + c_2\, t\, r^{r_1 t}$<br>$=$ exponential $\times$ monotonic function |

| | |
|---|---|
| Underdamped | Complex conjugate roots $\alpha \pm i\,\beta$<br>Negative discriminant<br>$x = e^{\alpha t}(c_1 \cos \beta t + c_2 \sin \beta t)$<br>$=$ exponential $\times$ harmonic oscillation |

**Bicycle trailer.** An auto tows a one–wheel trailer over a washboard road. Shown in Figure 19 is the trailer strut, which has a single coil spring and two dampeners. The mass $m$ includes the trailer and the bicycles.

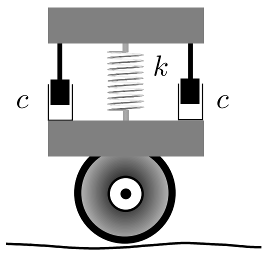


**Figure 19. A trailer strut with dampeners on a washboard road**

Suppose a washboard dirt road has about 2 full oscillations (2 bumps and 2 valleys) every 3 meters and a full oscillation has amplitude 6 centimeters. Let $s$ denote the horizontal distance along the road and let $\omega$ be the number of full oscillations of the roadway per unit length. The oscillation period is $2\pi/\omega$, therefore $2\pi/\omega = 3/2$ or $\omega = 4\pi/3$. A model for the road surface is

$$y = \frac{5}{100} \cos \omega s.$$

Let $x(t)$ denote the vertical elongation of the spring, measured from equilibrium. Newton's second law gives a force $F_1 = mx''(t)$ and the viscous damping force is $F_2 = 2cx'(t)$. The trailer elongates the spring by $x - y$, therefore the Hooke's force is $F_3 = k(x - y)$. The sum of the forces $F_1 + F_2 + F_3$ must be zero, which implies

$$mx''(t) + 2cx'(t) + k(x(t) - y(t)) = 0.$$

Write $s = vt$ where $v$ is the speedometer reading of the car in meters per second. The expanded differential equation is the forced damped spring-mass system equation

$$mx''(t) + 2cx'(t) + kx(t) = \frac{k}{20} \cos(4\pi vt/3).$$

The solution $x(t)$ of this model, with $x(0)$ and $x'(0)$ given, describes the vertical excursion of the trailer bed from the roadway. The **observed oscillations** of the trailer are modeled by the steady-state solution

$$x_{\mathrm{SS}}(t) = A \cos(4\pi vt/3) + B \sin(4\pi vt/3),$$

where $A$, $B$ are constants determined by the method of undetermined coefficients. From the physical data, the amplitude $\sqrt{A^2 + B^2}$ of this oscillation might be 6cm or larger.

## Exercises 5.7