Ch3. (Linear Systems and Matrices)

[25%] Ch3(a): Let $B$ be the invertible matrix given below. Write a formula for the second entry on the fourth row of the inverse matrix $B^{-1}$ as a quotient of two determinants. Do not evaluate any determinants.

$$B = \begin{pmatrix}
1 & 1 & -2 & 0 \\
1 & 1 & 0 & 0 \\
0 & -1 & 2 & 0 \\
1 & 0 & 0 & 3
\end{pmatrix}$$

[25%] Ch3(b): State the three possibilities for a linear system $Ax = b$. Determine which values of $k$ correspond to these three possibilities, for the system $Ax = b$ given in the display below.

$$A = \begin{pmatrix}
2 & 1 & -k \\
2 & k & -2 \\
0 & 0 & -k
\end{pmatrix}, \quad b = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

[25%] Ch3(c): Let $A$ be a $n \times n$ triangular matrix with diagonal entries $\sin(2\pi j/n)$ for $j = 1$ to $n$. Prove that $Ax = 0$ has a solution $x \neq 0$.

[25%] Ch3(d): Find the value of $x_3$ by Cramer’s Rule in the system $Cx = b$, given $C$ and $b$ below. Evaluate determinants by any hybrid method (triangular, swap, combo, multiply, cofactor). The use of $3 \times 3$ Sarrus’ rule is disallowed ($2 \times 2$ use is allowed).

$$C = \begin{pmatrix}
1 & 1 & -1 & 0 \\
1 & 3 & -2 & 1 \\
0 & 0 & 4 & 0 \\
0 & 0 & 2 & 1
\end{pmatrix}, \quad b = \begin{pmatrix} 0 \\ -1 \\ -1 \\ 1 \end{pmatrix}$$

If you solved (a), (b), (c) and (d), then go on to Ch4. Otherwise, continue here. Only four parts will be graded.

[25%] Ch3(e): Prove that if a $4 \times 4$ triangular matrix $C$ is invertible, then all diagonal entries of $C$ are nonzero.

Key on next page

Staple this page to the top of all Ch3 work. Submit one package per chapter.
A. Entry = \frac{\text{cofactor}(B_{2,4})}{\det(B)} = \begin{vmatrix} 1 & 1 & -2 \\ 0 & 1 & 2 \\ 1 & 0 & 0 \end{vmatrix} / \det(B), \text{ because } (-1)^{2+4} = 1

B. (1) No sol, (2) Unique sol, (3) infinitely many sols
   The sequence is k = 0 \Rightarrow \infty \text{- many sols} \begin{bmatrix} 2 & 1 & -k \\ 0 & k-1 & k-2 \\ 0 & 0 & k \end{bmatrix} = 0
   k \neq 0, k-1 \neq 0 \Rightarrow \text{unique sol}
   k = 1 \Rightarrow \text{no sol} (\text{since eq.})

C. Because \( j = n \), given diagonal entry \( a_{nn} = 0 \), then
   The triangular rule for determinants implies \( \det(A) = 0 \).

D. \( x_3 = \frac{\Delta_3}{\Delta} \), \( \Delta = 8 \), \( \Delta_3 = -2 \)

E. An invertible matrix has a nonzero determinant. A triangular matrix has a determinant equal to the product of its diagonal entries. Then all diagonal entries are nonzero.
Ch4. (Vector Spaces)

[50%] Ch4(a): Define V to be the set of all vectors x in $\mathbb{R}^4$ such that $x_1 = x_3$ and $x_1x_4 = 0$. Prove or disprove that V is a subspace of $\mathbb{R}^4$.

[50%] Ch4(b): Find a basis of fixed vectors in $\mathbb{R}^4$ for (1) the column space of the $4 \times 4$ matrix $A$ below [25%] and (2) the row space of the $4 \times 4$ matrix $A$ below [25%]. The two displayed bases must consist of columns of $A$ and rows of $A$, respectively.

\[
A = \begin{pmatrix}
1 & -1 & 1 & 0 \\
1 & -1 & -2 & -1 \\
2 & -2 & -1 & -1 \\
3 & -3 & -3 & -2
\end{pmatrix}
\]

If you did both (a) and (b), then go on to Ch5. Otherwise, continue here.

[25%] Ch4(c): State an RREF test and also a determinant test to detect the independence or dependence of fixed vectors $v_1, v_2, v_3$ [10%]. Apply one of the tests to the vectors below [10%]. Report independent or dependent [5%].

\[
v_1 = \begin{pmatrix}
-1 \\
1 \\
2 \\
1
\end{pmatrix}, \quad v_2 = \begin{pmatrix}
3 \\
0 \\
1 \\
1
\end{pmatrix}, \quad v_3 = \begin{pmatrix}
4 \\
-1 \\
-1 \\
1
\end{pmatrix}
\]

[25%] Ch4(d): Let $v_1, v_2$ be a basis for a subspace $S$ of a vector space $V$. Let $w_1, w_2$ be any other proposed basis of $S$. State a test which can decide whether or not the two bases are equivalent.

A. Not a subspace, because $\overline{v_1} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\overline{v_2} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ are in $V$, but $\overline{v_1} + \overline{v_2} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is not in $V$.

B. Find $\text{rref}(A)$, pivots $= 1, 3$. So the column space basis = \{col(A, 1), col(A, 3)\}

Find $\text{rref}(A^T)$, pivots $= 1, 2$. So the row space basis = \{row(A, 1), row(A, 2)\}

C. Thm1. $v_1, v_2, v_3$ independent $\iff \text{rank}(\text{aug}(v_1, v_2, v_3)) = 3$

Thm2. If $A = \text{aug}(v_1, v_2, v_3)$ is $3 \times 3$, then $v_1, v_2, v_3$ independent $\iff \text{det}(A) \neq 0$.

Because $\text{aug}(v_1, v_2, v_3)$ has rank $= 3$, then independent.

D. Let $B = \text{aug}(v_1, v_2), C = \text{aug}(w_1, w_2), D = \text{aug}(v_1, v_2, w_1, w_2)$. Then the bases are equivalent if $B, C, D$ all have rank 2.

Staple this page to the top of all Ch4 work. Submit one package per chapter.
Ch5. (Linear Equations of Higher Order)

[10%] Ch5(a): Find the general solution, given characteristic equation

\[ 4r^2 + 17r + 4 = 0. \]

[15%] Ch5(b): Find the general solution, given characteristic equation

\[ (r^2 + 5r)^3(r^4 - 25r^2) = 0. \]

[15%] Ch5(c): Find the general solution, given characteristic equation

\[ (r^2 + 2r + 2)^3(r^2 + 16)^2 = 0. \]

[40%] Ch5(d): Assume a sixth order constant-coefficient differential equation has characteristic equation \( r^2(r^2 + 4)^2 = 0. \) Suppose the right side of the differential equation is

\[ f(x) = x^2(1 + 2e^x) + 3x \sin 2x + x \cos 2x. \]

Determine the corrected trial solution for \( y_p \) according to the method of undetermined coefficients. Do not evaluate the undetermined coefficients!

[20%] Ch5(e): Find the steady-state periodic solution for the equation

\[ x'' + 4x' + 6x = \cos(3t). \]

\[ y = \text{linear combination } e^{-\gamma t}, e^{-\gamma t}, \text{ because it factors } (4\gamma + 1)(\gamma + 1)(\gamma + 5) = 0 \]

\[ y'' + 5(\gamma + 5)y' + (\gamma + 5)^2y = 0, \quad \gamma = 0, 5, 5, 5, 5, 5, 5 \]

\[ y = \text{linear combination } e^{\gamma t}, e^{\gamma t}, e^{\gamma t}, e^{\gamma t}, e^{\gamma t}, e^{\gamma t}, e^{\gamma t} \]

\[ y = \text{linear combination } e^{\gamma t}, e^{\gamma t}, e^{\gamma t}, e^{\gamma t}, e^{\gamma t}, e^{\gamma t}, e^{\gamma t} \]

\[ y = \text{linear combination } e^{\gamma t}, e^{\gamma t}, e^{\gamma t}, e^{\gamma t}, e^{\gamma t}, e^{\gamma t}, e^{\gamma t} \]

\[ y = \text{linear combination } e^{\gamma t}, e^{\gamma t}, e^{\gamma t}, e^{\gamma t}, e^{\gamma t}, e^{\gamma t}, e^{\gamma t} \]

\[ y_{ss} = -\frac{1}{51} \cos(3t) + \frac{1}{51} \sin(3t) \]

Staple this page to the top of all Ch5 work. Submit one package per chapter.
Ch6. (Eigenvalues and Eigenvectors)

☐ [25%] Ch6(a): Find the eigenvalues of the matrix

\[
A = \begin{pmatrix}
0 & 1 & -1 & 0 \\
-1 & 0 & -2 & 1 \\
0 & 0 & 4 & 1 \\
0 & 0 & 1 & 2
\end{pmatrix}
\]

☐ [10%] Ch6(b): Let \( A \) be a \( 2 \times 2 \) matrix with eigenpairs

\[
\begin{pmatrix} 2, \begin{pmatrix} 1 \\ -1 \end{pmatrix} \end{pmatrix}, \quad \begin{pmatrix} 3, \begin{pmatrix} -1 \\ 2 \end{pmatrix} \end{pmatrix}.
\]

Display Fourier’s model for the matrix \( A \).

☐ [15%] Ch6(c): Find a \( 2 \times 2 \) matrix \( A \) with eigenpairs

\[
\begin{pmatrix} -1, \begin{pmatrix} -1 \\ 2 \end{pmatrix} \end{pmatrix}, \quad \begin{pmatrix} -2, \begin{pmatrix} 1 \\ 2 \end{pmatrix} \end{pmatrix}.
\]

☐ [50%] Ch6(d): Test the matrix \( A \) below to see if it is diagonalizable. If it is, then display the matrix packages \( P \) and \( D \) of eigenvectors and eigenvalues.

\[
A = \begin{pmatrix}
2 & 3 & -9 \\
0 & 8 & -18 \\
0 & 3 & -7
\end{pmatrix}
\]

\( a \) \( i_1, i_2, 3 + \sqrt{2}, 3 - \sqrt{2} \)

\( b \) \( A(c_1 \begin{pmatrix} -1 \end{pmatrix} + c_2 \begin{pmatrix} -1 \end{pmatrix}) = 2c_1 \begin{pmatrix} -1 \end{pmatrix} + 3c_2 \begin{pmatrix} -1 \end{pmatrix} \)

\( c \) \( A = PDP^{-1} = \begin{pmatrix} -3/2 & -1/4 \\
-1 & -3/2
\end{pmatrix} \)

\( d \) \( \lambda_1 = -1, \lambda_2 = 2 (\text{multiplicity } 2) \).

\[
P = \begin{pmatrix}
1 & 1 & 0 \\
2 & 0 & 1
\end{pmatrix}, \quad D = \begin{pmatrix}
-1 & 0 & 0 \\
0 & 2 & 0 \\
0 & 0 & 2
\end{pmatrix}.
\]

Staple this page to the top of all Ch6 work. Submit one package per chapter.
Ch7. (Linear Systems of Differential Equations)

- [50%] Ch7(a): Apply the eigenanalysis method to solve the system $x' = Ax$, given
  \[
  A = \begin{pmatrix}
  0 & 1 & 1 \\
  1 & 0 & 1 \\
  0 & 0 & -1 
  \end{pmatrix}
  \]

- [25%] Ch7(b): Give an example of a $2 \times 2$ real matrix $A$ for which Fourier's model is not valid. Then display the general solution $x(t)$ of $x' = Ax$.

- [25%] Ch7(c): Consider a $2 \times 2$ system $x' = Ax$. Assume $A$ has complex eigenvalues $\lambda = -2 \pm \sqrt{2}i$. Prove that all solutions satisfy $\lim_{t \to \infty} |x(t)| = 0$.

If you solved (a), (b), (c), then go on to Ch10. Otherwise, continue here. You may select either (a) or (d) but not both. Only 3 parts will be graded. The maximum score is 75 if you select (d).

- [25%] Ch7(d): A $3 \times 3$ real matrix $A$ has all eigenvalues equal to zero and corresponding eigenvectors
  \[
  \begin{pmatrix}
  0 \\ 1 \\ 1 
  \end{pmatrix},
  \begin{pmatrix}
  0 \\ -5 \\ 1 
  \end{pmatrix},
  \begin{pmatrix}
  1 \\ 1 \\ 2 
  \end{pmatrix}.
  \]

Find the general solution of the differential equation $x' = Ax$.

- (a) $\ddot{x}(t) = c_1 e^{t} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + c_2 e^{t} \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix} + c_3 e^{-t} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad \lambda = 1, -1, -1$

- (b) $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ Eigenvalues $= 0, 0$; one eigenvector $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$

\[
\begin{cases}
  x' = y \\
  y' = 0
\end{cases}
\]

has solution
\[
\begin{cases}
  x = c_1 + c_2 t \\
  y = c_2
\end{cases}
\]

- Fourier's model holds. Then $\ddot{x} = c_1 e^{-2t} \cos \sqrt{2}t \ddot{v}_1 + c_2 e^{-2t} \sin \sqrt{2}t \ddot{v}_2$ for some real vectors $\ddot{v}_1, \ddot{v}_2$. Because $e^{-2t} \to 0$ as $t \to \infty$, then $|x(t)| \to 0$ as $t \to \infty$.

- (d) $\ddot{x} = c_1 \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ -5 \\ 1 \end{pmatrix} + c_3 \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$ because $e^{0t} = 1$.

Staple this page to the top of all Ch7 work. Submit one package per chapter.
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Ch10. (Laplace Transform Methods)
It is assumed that you have memorized the basic 4-item Laplace integral table and know the 6 basic rules for Laplace integrals. No other tables or theory are required to solve the problems below. If you don’t know a table entry, then leave the expression unevaluated for partial credit.

\[ (35\%) \text{ Ch10(a): Apply Laplace's method to the system to find a formula for } \mathcal{L}(y(t)). \text{ Find a } 2 \times 2 \text{ system for } \mathcal{L}(x), \mathcal{L}(y) [20\%]. \text{ Solve it only for } \mathcal{L}(y) [15\%]. \text{ Do not solve for } x(t) \text{ or } y(t)! \]

\[
\begin{align*}
x'' &= -y, \\
y'' &= 3x, \\
x(0) &= 0, \quad x'(0) = 0, \\
y(0) &= 0, \quad y'(0) = -1.
\end{align*}
\]

\[ (35\%) \text{ Ch10(b): Solve for } f(t), \text{ given} \]

\[
\mathcal{L}(f(t)) = \left( \frac{d}{ds} (\mathcal{L}(\sin 2t)) \right) \bigg|_{s \to s+3} + \frac{s + 1}{(s-2)^2} - \mathcal{L} \left( \frac{d}{dt} (t^2 \sin 2t) \right)
\]

\[ (30\%) \text{ Ch10(c): Find } f(t) \text{ by partial fraction methods, given} \]

\[
\mathcal{L}(f(t)) = \frac{1}{s^2 - 2s} + \frac{2s^2 - 6}{(s^2 - 1)(s-1)}.
\]

If you solved (a), (b) and (c), then you have 100\%. To select (d), unmark one of the previous three. Only three parts will be graded. If you did not solve (a) or (b), then the maximum score is 95.

\[ (30\%) \text{ Ch10(d): Apply Laplace's method to find a formula for } \mathcal{L}(x(t)). \text{ Do not solve for } x(t)! \]

Document steps by reference to tables and rules.

\[
x^{iv} - 4x'' = e^{2t}(5 + 4t + 3 \sin 2t), \quad x(0) = x'(0) = x''(0) = 0, \quad x'''(0) = -1.
\]

\begin{enumerate}
\item \( \mathcal{L}(y) = \begin{vmatrix} s^2 & 0 \\ 3 & 1 \end{vmatrix} = \frac{-5^2}{s^4 + 3} \)
\item \( f(t) = -\frac{1}{2} e^{2t} \sin 2t + e^{2t} + 3t e^{2t} - 2t \sin 2t - 2t^2 \cos 2t \)
\item \( \frac{1}{s} + \frac{1}{s-2} + \frac{3}{s+1} + \frac{2}{s-1} \)
\item \( \mathcal{L}(x(t)) = \frac{\text{top}}{\text{bot}} \quad \text{top} = -1 + \frac{5}{s-2} + \frac{4}{(s-2)^2} + \frac{6}{(s-2)^2 + 4}, \quad \text{bot} = s - 4s^2 \)
\end{enumerate}

Staple this page to the top of all Ch10 work. Submit one package per chapter.