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Differential Equations and Linear Algebra 2250-4

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Ch3. (Linear Systems and Matrices)

☐ [25%] Ch3(a): Let B be the invertible matrix given below. Write a formula for the **second entry on the fourth row** of the inverse matrix B^{-1} as a quotient of two determinants. **Do not evaluate any determinants**.

$$B = \begin{pmatrix} 1 & 1 & -2 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & -1 & 2 & 0 \\ 1 & 0 & 0 & 3 \end{pmatrix}$$

☐ [25%] Ch3(b): State the three possibilities for a linear system $A\mathbf{x} = \mathbf{b}$. Determine which values of k correspond to these three possibilities, for the system $A\mathbf{x} = \mathbf{b}$ given in the display below.

$$A = \begin{pmatrix} 2 & 1 & -k \\ 2 & k & -2 \\ 0 & 0 & -k \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

☐ [25%] Ch3(c): Let A be a $n \times n$ triangular matrix with diagonal entries $\sin(2\pi j/n)$ for $j = 1$ to n. Prove that $A\mathbf{x} = \mathbf{0}$ has a solution $\mathbf{x} \neq \mathbf{0}$.

☐ [25%] Ch3(d): Find the value of x_3 by Cramer's Rule in the system $C\mathbf{x} = \mathbf{b}$, given C and $\mathbf{b}$ below. Evaluate determinants by any hybrid method (triangular, swap, combo, multiply, cofactor). The use of 3×3 Sarrus' rule is disallowed (2×2 use is allowed).

$$C = \begin{pmatrix} 1 & 1 & -1 & 0 \\ 1 & 3 & -2 & 1 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 2 & 1 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 0 \\ -1 \\ -1 \\ 1 \end{pmatrix}$$

If you solved (a), (b), (c) and (d), then go on to Ch4. Otherwise, continue here. Only four parts will be graded.

☐ [25%] Ch3(e): Prove that if a 4×4 triangular matrix C is invertible, then all diagonal entries of C are nonzero.

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(a) Entry $= \dfrac{\text{cofactor}(B,2,4)}{\det(B)} = \begin{vmatrix} 1 & 1 & -2 \\ 0 & -1 & 2 \\ 1 & 0 & 0 \end{vmatrix} \Big/ \det(B)$, because $(-1)^{2+4} = 1$

(b) (1) No sol, (2) unique sol, (3) ∞-many sols

one frame in the sequence is

$$\left(\begin{array}{ccc|c} 2 & 1 & -k & 0 \\ 0 & k-1 & k-2 & 1 \\ 0 & 0 & k & 0 \end{array}\right)$$

$k=0 \Rightarrow \infty$-many sols

$k \neq 0, k-1 \neq 0 \Rightarrow$ unique sol

$k=1 \Rightarrow$ No sol (signal eq.)

(c) Because $j=n$ given diagonal entry $\sin(2\pi)=0$, then the triangular rule for determinants implies $\det(A)=0$.

(d) $x_3 = \dfrac{\Delta_3}{\Delta}$, $\Delta = 8$, $\Delta_3 = -2$

(e) An invertible matrix has nonzero determinant. A triangular matrix has determinant equal to the product of the diagonal entries. Then all diagonal entries are nonzero.

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Ch4. (Vector Spaces)

☐ [50%] Ch4(a): Define V to be the set of all vectors $\mathbf{x}$ in $\mathcal{R}^4$ such that $x_1 = x_3$ and $x_1 x_4 = 0$. Prove or disprove that V is a subspace of $\mathcal{R}^4$.

☐ [50%] Ch4(b): Find a basis of fixed vectors in $\mathcal{R}^4$ for (1) the column space of the 4×4 matrix A below [25%] and (2) the row space of the 4×4 matrix A below [25%]. The two displayed bases **must** consist of columns of A and rows of A, respectively.

$$A = \begin{pmatrix} 1 & -1 & 1 & 0 \\ 1 & -1 & -2 & -1 \\ 2 & -2 & -1 & -1 \\ 3 & -3 & -3 & -2 \end{pmatrix}$$

If you did both (a) and (b), then go on to Ch5. Otherwise, continue here.

☐ [25%] Ch4(c): State an RREF test and also a determinant test to detect the independence or dependence of fixed vectors $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$ [10%]. Apply one of the tests to the vectors below [10%]. Report **independent** or **dependent** [5%].

$$\mathbf{v}_1 = \begin{pmatrix} -1 \\ 1 \\ 2 \\ 1 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} 3 \\ 0 \\ 1 \\ 1 \end{pmatrix}, \quad \mathbf{v}_3 = \begin{pmatrix} 4 \\ -1 \\ -1 \\ 1 \end{pmatrix}.$$

☐ [25%] Ch4(d): Let $\mathbf{v}_1$, $\mathbf{v}_2$ be a basis for a subspace S of a vector space V. Let $\mathbf{w}_1$, $\mathbf{w}_2$ be any other proposed basis of S. State a test which can decide whether or not the two bases are equivalent.

(a) Not a subspace, because $\vec{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}$ and $\vec{v}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}$ are in V, but $\vec{v}_1 + \vec{v}_2 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$ is not in V.

(b) Find rref(A), pivots = 1, 3. So the column space basis = {col(A,1), col(A,3)}
Find rref(A^T), pivots = 1, 2. So the row space basis = {row(A,1), row(A,2)}.

(c) Thm 1. v_1, v_2, v_3 independent $\Leftrightarrow$ rank(aug(v_1, v_2, v_3)) = 3
Thm 2. If A = aug(v_1, v_2, v_3) is 3×3, then v_1, v_2, v_3 independent $\Leftrightarrow \det(A) \neq 0$.
Because aug(v_1, v_2, v_3) has rank = 3, then independent.

(d) Let B = aug(v_1, v_2), C = aug(w_1, w_2), D = aug(v_1, v_2, w_1, w_2).
Then the bases are equivalent if B, C, D all have rank 2.

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Ch5. (Linear Equations of Higher Order)

☐ [10%] Ch5(a): Find the general solution, given characteristic equation

$$4r^2 + 17r + 4 = 0.$$

☐ [15%] Ch5(b): Find the general solution, given characteristic equation

$$(r^2 + 5r)^3(r^4 - 25r^2) = 0.$$

☐ [15%] Ch5(c): Find the general solution, given characteristic equation

$$(r^2 + 2r + 2)^3(r^2 + 16)^2 = 0.$$

☐ [40%] Ch5(d): Assume a sixth order constant-coefficient differential equation has characteristic equation $r^2(r^2+4)^2 = 0$. Suppose the right side of the differential equation is

$$f(x) = x^2(1 + 2e^x) + 3x \sin 2x + x \cos 2x$$

Determine the **corrected** trial solution for y_p according to the method of undetermined coefficients. **Do not evaluate** the undetermined coefficients!

☐ [20%] Ch5(e): Find the steady-state periodic solution for the equation

$$x'' + 4x' + 6x = \cos(3t).$$

(a) $y =$ linear combination of $e^{-x/4}, e^{-4x}$ because it factors $(4r+1)(r+4)=0$

(b) $r^5(r+5)^4(r-5)=0$, $r = 0,0,0,0,0,-5,-5,-5,-5,5$
$y =$ l.c. of atoms in $L = \{1, x, x^2, x^3, x^4, e^{-5x}, xe^{-5x}, x^2e^{-5x}, x^3e^{-5x}, e^{5x}\}$

(c) Roots $= -1\pm i, -1\pm i, -1\pm i, 4i, 4i, -4i, -4i$
$y =$ l.c. of atoms in $L = \{e^{-x}\cos x, xe^{-x}\cos x, x^2e^{-x}\cos x, e^{-x}\sin x,$
$xe^{-x}\sin x, x^2e^{-x}\sin x, \cos 4x, x\cos 4x,$
$\sin 4x, x\sin 4x\}$.

(d) $y = (d_1 + d_2 x + d_3 x^2)x^2$
$+ (d_4 e^x + d_5 xe^x + d_6 x^2 e^x)$
$+ (d_7 \cos 2x + d_8 \sin 2x + d_9 x\cos 2x + d_{10} x \sin 2x)x^2$

(e) $y_{ss} = -\frac{1}{51}\cos(3t) + \frac{4}{51}\sin(3t)$

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Ch6. (Eigenvalues and Eigenvectors)

☐ [25%] Ch6(a): Find the eigenvalues of the matrix

$$A = \begin{pmatrix} 0 & 1 & -1 & 0 \\ -1 & 0 & -2 & 1 \\ 0 & 0 & 4 & 1 \\ 0 & 0 & 1 & 2 \end{pmatrix}$$

☐ [10%] Ch6(b): Let A be a 2×2 matrix with eigenpairs

$$\left(2, \begin{pmatrix} 1 \\ -1 \end{pmatrix}\right), \quad \left(3, \begin{pmatrix} -1 \\ 2 \end{pmatrix}\right).$$

Display Fourier's model for the matrix A.

☐ [15%] Ch6(c): Find a 2×2 matrix A with eigenpairs

$$\left(-1, \begin{pmatrix} -1 \\ 2 \end{pmatrix}\right), \quad \left(-2, \begin{pmatrix} 1 \\ 2 \end{pmatrix}\right).$$

☐ [50%] Ch6(d): Test the matrix A below to see if it is diagonalizable. If it is, then display the matrix packages P and D of eigenvectors and eigenvalues.

$$A = \begin{pmatrix} 2 & 3 & -9 \\ 0 & 8 & -18 \\ 0 & 3 & -7 \end{pmatrix}$$

(a) $i, -i, \; 3+\sqrt{2}, \; 3-\sqrt{2}$

(b) $A\left(c_1\begin{pmatrix} 1 \\ -1 \end{pmatrix} + c_2\begin{pmatrix} -1 \\ 2 \end{pmatrix}\right) = 2c_1\begin{pmatrix} 1 \\ -1 \end{pmatrix} + 3c_2\begin{pmatrix} -1 \\ 2 \end{pmatrix}$

(c) $A = PDP^{-1} = \begin{pmatrix} -3/2 & -1/4 \\ -1 & -3/2 \end{pmatrix}$

(d) $\lambda_1 = -1, \; \lambda_2 = 2, 2$ (multiplicity 2).

$P = \begin{pmatrix} 1 & 1 & 0 \\ 2 & 0 & 3 \\ 1 & 0 & 1 \end{pmatrix}, \; D = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$

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Ch7. (Linear Systems of Differential Equations)

☐ [50%] Ch7(a): Apply the eigenanalysis method to solve the system $\mathbf{x}' = A\mathbf{x}$, given

$$A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 0 & -1 \end{pmatrix}$$

☐ [25%] Ch7(b): Give an example of a 2×2 real matrix A for which Fourier's model is not valid. Then display the general solution $\mathbf{x}(t)$ of $\mathbf{x}' = A\mathbf{x}$.

☐ [25%] Ch7(c): Consider a 2×2 system $\mathbf{x}' = A\mathbf{x}$. Assume A has complex eigenvalues $\lambda = -2 \pm \sqrt{2}i$. Prove that all solutions satisfy $\lim_{t\to\infty} |\mathbf{x}(t)| = 0$.

If you solved (a), (b), (c), then go on to Ch10. Otherwise, continue here. You may select either (a) or (d) but not both. Only 3 parts will be graded. The maximum score is 75 if you select (d).

☐ [25%] Ch7(d): A 3×3 real matrix A has all eigenvalues equal to zero and corresponding eigenvectors

$$\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ -5 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}.$$

Find the general solution of the differential equation $\mathbf{x}' = A\mathbf{x}$.

(a) $\vec{x}(t) = c_1 e^{t} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + c_2 e^{-t} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} + c_3 e^{-t} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$ $\lambda = 1, -1, -1$

(b) $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ Eigenvalues $= 0, 0$; one eigenvector $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$\begin{cases} x' = y \\ y' = 0 \end{cases}$ has sol $\begin{cases} x = c_1 + c_2 t \\ y = c_2 \end{cases}$

(c) Fourier's model holds. Then $\vec{X} = c_1 e^{-2t} \cos\sqrt{2}t \, \vec{V}_1 + c_2 e^{-2t} \sin\sqrt{2}t \, \vec{V}_2$ for some real vectors $\vec{V}_1, \vec{V}_2$. Because $e^{-2t} \to 0$ as $t \to \infty$, Then $|X(t)| \to 0$ as $t \to \infty$.

(d) $\vec{x} = c_1 \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ -5 \\ 1 \end{pmatrix} + c_3 \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$ because $e^{0t} = 1$.

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Ch10. (Laplace Transform Methods)

It is assumed that you have memorized the basic 4-item Laplace integral table and know the 6 basic rules for Laplace integrals. No other tables or theory are required to solve the problems below. If you don't know a table entry, then leave the expression unevaluated for partial credit.

☐ [35%] Ch10(a): Apply Laplace's method to the system to find a formula for $\mathcal{L}(y(t))$. Find a 2×2 system for $\mathcal{L}(x)$, $\mathcal{L}(y)$ [20%]. Solve it **only** for $\mathcal{L}(y)$ [15%]. Do not solve for $x(t)$ or $y(t)$!

$$\begin{aligned} &x'' = -y, \\ &y'' = 3x, \\ &x(0) = 0, \quad x'(0) = 0, \\ &y(0) = 0, \quad y'(0) = -1. \end{aligned}$$

☐ [35%] Ch10(b): Solve for $f(t)$, given

$$\mathcal{L}(f(t)) = \left(\frac{d}{ds}\left(\mathcal{L}(\sin 2t)\right)\right)\bigg|_{s\to(s+3)} + \frac{s+1}{(s-2)^2} - \mathcal{L}\left(\frac{d}{dt}\left(t^2\sin 2t\right)\right)$$

☐ [30%] Ch10(c): Find $f(t)$ by partial fraction methods, given

$$\mathcal{L}(f(t)) = \frac{1-s}{s^2-2s} + \frac{2s^2-6}{(s^2-1)(s-1)}.$$

If you solved (a), (b) and (c), then you have 100%. To select (d), unmark one of the previous three. Only three parts will be graded. If you did not solve (a) or (b), then the maximum score is 95.

☐ [30%] Ch10(d): Apply Laplace's method to find a formula for $\mathcal{L}(x(t))$. **Do not** solve for $x(t)$! Document steps by reference to tables and rules.

$$x^{iv} - 4x'' = e^{2t}(5 + 4t + 3\sin 2t), \quad x(0) = x'(0) = x''(0) = 0, \quad x'''(0) = -1.$$

(a) $\mathcal{L}(y) = \dfrac{\begin{vmatrix} s^2 & 0 \\ 3 & 1 \end{vmatrix}}{\begin{vmatrix} s^2 & 1 \\ 3 & -s^2 \end{vmatrix}} = \dfrac{-s^2}{s^4+3}$

(b) $f(t) = -t\,e^{-3t}\sin 2t + e^{2t} + 3te^{2t} - 2t\sin 2t - 2t^2\cos 2t$

(c) $\dfrac{1/2}{s} + \dfrac{-1/2}{s-2} + \dfrac{-1}{s+1} + \dfrac{3}{s-1} + \dfrac{-2}{(s-1)^2}$ $\quad f(t) = -\frac{1}{2} - \frac{1}{2}e^{2t} - e^{-t} + 3e^{t} - 2te^{t}$

(d) $\mathcal{L}(x(t)) = \dfrac{\text{top}}{\text{bot}}$ $\quad \text{top} = -1 + \dfrac{5}{s-2} + \dfrac{4}{(s-2)^2} + \dfrac{6}{(s-2)^2+4}$, $\text{bot} = s^4 - 4s^2$