KEY

Differential Equations and Linear Algebra

2250-2 10:10am 13 December 2006

Instructions. The time allowed is 120 minutes. The examination consists of six problems, one for each of chapters 3, 4, 5, 6, 7, 10, each problem with multiple parts. A chapter represents 20 minutes on the final exam.

The final exam counts as two midterm exams. For example, if exam scores earned were 90, 91, 92 and the final exam score is 89, then the exam average for the course is

$$\frac{90+91+92+89+89}{5}.$$

Each problem represents several textbook problems numbered (a), (b), (c), $\cdots$. Choose the problems to be graded by check-mark $\boxed{\text{X}}$; the credits should add to 100. Each chapter (Ch3, Ch4, Ch5, Ch6, ch7, Ch10) adds at most 100 towards the maximum final exam score of 600. The final exam score is reported as a percentage 0 to 100, which is the sum of the scores earned on six chapters divided by 600, then converted to a percentage.

Calculators, books, notes and computers are not allowed.

Details count. Solutions requiring details are considered incomplete. Less than full credit is earned, in this case, for an answer only.

Answer checks are not expected or required. First drafts are expected, not complete presentations.

Please submit **exactly six** separately stapled packages of problems, one package per chapter.

Please discard this page or keep it for your records.

Name KEY

Differential Equations and Linear Algebra 2250-2

Final Exam 10:10am 13 December 2006

Ch3. (Linear Systems and Matrices)

☐ [25%] Ch3(a): Let B be the invertible matrix given below, where $\boxed{?}$ means the value of the entry does not affect the answer to this problem. The second matrix C is the adjugate (or adjoint) of B. Find the value of $\det(B)$.

$$B = \begin{pmatrix} ? & ? & -2 & ? \\ ? & ? & 0 & ? \\ 0 & -1 & 2 & 0 \\ ? & ? & 0 & ? \end{pmatrix}, \quad C = \begin{pmatrix} 6 & 0 & 6 & 0 \\ -6 & 6 & -6 & 0 \\ -3 & 3 & 0 & 0 \\ -2 & 0 & -2 & 2 \end{pmatrix}$$

☐ [25%] Ch3(b): State the three possibilities for a linear system $A\mathbf{x} = \mathbf{b}$. Determine which values of k correspond to these three possibilities, for the system $A\mathbf{x} = \mathbf{b}$ given in the display below.

$$A = \begin{pmatrix} 2 & 1 & k \\ 2 & -k & -2 \\ 0 & 0 & k \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$$

☐ [25%] Ch3(c): Assume A is an $n \times n$ matrix. A theorem says that A invertible implies $A\mathbf{x} = \mathbf{b}$ has a unique solution. State two different theorems, each with the same conclusion: $A\mathbf{x} = \mathbf{b}$ has a unique solution.

☐ [25%] Ch3(d): Find the value of x_2 by Cramer's Rule in the system $C\mathbf{x} = \mathbf{b}$, given C and $\mathbf{b}$ below. Evaluate determinants by any hybrid method (triangular, swap, combo, multiply, cofactor). The use of 2×2 Sarrus' rule is allowed (3×3 use is disallowed).

$$C = \begin{pmatrix} 1 & 1 & -1 & 0 \\ 1 & 3 & -2 & 1 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 2 & 1 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 0 \\ -1 \\ -1 \\ 1 \end{pmatrix}$$

If you solved (a), (b), (c) and (d), then go on to Ch4. Otherwise, continue here. Only four parts will be graded.

☐ [25%] Ch3(e): Give an example of a 4×3 matrix such that $A\mathbf{x} = \mathbf{0}$ has a unique solution.

(a) $BC = CB = \det(B)I \Rightarrow \text{row}(B,3)\text{col}(C,3) = \det(B) \Rightarrow \boxed{\det B = 6}$

(b) Unique sol $k(k+1) \neq 0$, No Solution $k=-1$, ∞-Many Solutions $k=0$

(c) 1. $A\vec{x}=\vec{0}$ has sol only $\vec{x}=\vec{0}$. 2. $\det(A) \neq 0$. 3. $\text{rank}(\text{aug}(A,b))=3$
4. $\text{rref}(A)=I$ 5. 3 lead variables 6. 0 free variables.

(d) $x_2 = \Delta_2/\Delta = -11/8$

(e) $A = \begin{pmatrix} 1&0&0 \\ 0&1&0 \\ 0&0&1 \\ 0&0&0 \end{pmatrix}$ 3 lead vars $\Rightarrow$ unique sol $\vec{x}=\vec{0}$.

Staple this page to the top of all Ch3 work. Submit one package per chapter.

Name KEY

Differential Equations and Linear Algebra 2250-2

Final Exam 10:10am 13 December 2006

Ch4. (Vector Spaces)

☐ [50%] Ch4(a): Define S to be the set of all vectors $\mathbf{x}$ in $\mathcal{R}^4$ such that $x_1 = x_3$ and $e^{x_1 - 2x_4} = 1$. Prove or disprove that S is a subspace of $\mathcal{R}^4$.

☐ [50%] Ch4(b): Define the 4×4 matrix A by the display below. Find a basis of fixed vectors in $\mathcal{R}^4$ for (1) the column space of A [25%] and (2) the row space A [25%]. The two displayed bases **must** consist of columns of A and rows of A, respectively.

$$A = \begin{pmatrix} 1 & -1 & 1 & 0 \\ 3 & -3 & -3 & -2 \\ 1 & -1 & -2 & -1 \\ 2 & -2 & -1 & -1 \end{pmatrix}$$

If you did both (a) and (b), then 100% has been marked – go on to Ch5. Otherwise, unmark one of (a), (b), then complete (c), (d).

☐ [25%] Ch4(c): State (1) an RREF or rank test and (2) a determinant test to detect the independence or dependence of fixed vectors $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$ [10%]. Apply one of the tests to the vectors below [10%]. Report **independent** or **dependent** [5%].

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \quad \mathbf{v}_3 = \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}.$$

☐ [25%] Ch4(d): Let $\mathbf{v}_1$, $\mathbf{v}_2$ be a basis for a subspace S of vector space $V = \mathcal{R}^n$. Let $\mathbf{w}_1$, $\mathbf{w}_2$ be any other proposed basis of S. State a test which can decide whether or not the two bases are equivalent.

(a) Equations $\begin{cases} x_1 - x_3 = 0 \\ x_1 - 2x_4 = 0 \end{cases}$ because $e^u = 1 \Leftrightarrow u = 0$. Let $A = \begin{pmatrix} 1 & 0 & -1 & 0 \\ 1 & 0 & 0 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$.
Then $S = \{\vec{x} : A\vec{x} = \vec{0}\}$. Apply Thm 2, 4.2 in E&P.

(b) $\mathrm{rref}(A) = \begin{pmatrix} 1 & -1 & 0 & * \\ 0 & 0 & 1 & * \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ $\quad \mathrm{rref}(A^T) = \begin{pmatrix} 1 & 0 & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ $\quad$ col(A,1), col(A,3)
row(A,1), row(A,2)

(c) Test 1: rref(aug(v_1, v_2, v_3)) has 3 lead vars $\Leftrightarrow$ independent
rank(aug(v_1, v_2, v_3)) = 3 $\Leftrightarrow$ independent
Test 2: det(aug(v_1, v_2, v_3)) $\neq 0$ $\Leftrightarrow$ independent
apply Test 2, det = 1 $\Rightarrow$ independent.

(d) B = aug(v_1, v_2), C = aug(w_1, w_2), D = aug(v_1, v_2, w_1, w_2)
Equivalent Bases $\Leftrightarrow$ rank(B) = rank(C) = rank(D) = 2

Staple this page to the top of all Ch4 work. Submit one package per chapter.

Name KEY

Differential Equations and Linear Algebra 2250-2

Final Exam 10:10am 13 December 2006

Ch5. (Linear Equations of Higher Order)

☐ [10%] Ch5(a): Find the general solution of the differential equation

$$y'' + \frac{17}{4}y' + y = 0.$$

☐ [20%] Ch5(b): Find the homogeneous differential equation general solution, given characteristic equation

$$(r^2 + 5r)^3(r^4 - 25r^2)(r^2 + 2r + 2)^2 = 0.$$

☐ [10%] Ch5(c): Given a damped spring-mass system $mx''(t) + cx'(t) + kx(t) = 0$ with $m = 15$, $c = 17$ and $k = 4$, classify the answer as over-damped, critically damped or under-damped. Please, **do not solve** the differential equation!

☐ [40%] Ch5(d): Assume a seventh order constant-coefficient differential equation has characteristic equation $r(r^2 + 4)(r^2 - 16) = 0$. Suppose the right side of the differential equation is

$$f(x) = x(x + 2e^x) + 3\sin 2x + 4x\cos 2x$$

Determine the **corrected** trial solution for y_p according to the method of undetermined coefficients. **Do not evaluate** the undetermined coefficients!

☐ [20%] Ch5(e): Find the steady-state periodic solution for the equation

$$x'' + 4x' + 20x = \cos(5t).$$

(a) $y = c_1 e^{-x/4} + c_2 e^{-4x}$

(b) y = linear combination of atoms in L

$L = \{1, x, x^2, x^3, x^4, e^{-5x}, xe^{-5x}, x^2e^{-5x}, x^3e^{-5x}, e^{5x}, e^{-x}\cos x, e^{-x}\sin x, xe^{-x}\cos x, xe^{-x}\sin x\}$

(c) Discriminant $= 17^2 - (15)(16) > 0 \Rightarrow$ over damped

(d) $L = \{1, \cos 2x, \sin 2x, e^{4x}, e^{-4x}\}$

$y = (d_1 + d_2x + d_3x^2)x + (d_4e^x + d_5xe^x) + (d_6\cos 2x + d_7\sin 2x + xd_8\cos 2x + xd_9\sin 2x)x$

(e) $y = d_1\cos 5t + d_2\sin 5t$, $d_1 = \frac{-1}{85}$, $d_2 = \frac{4}{85}$

Staple this page to the top of all Ch5 work. Submit one package per chapter.

Name KEY

Differential Equations and Linear Algebra 2250-2

Final Exam 10:10am 13 December 2006

Ch6. (Eigenvalues and Eigenvectors)

☐ [25%] Ch6(a): Find the eigenvalues of the matrix

$$A = \begin{pmatrix} 0 & 4 & -1 & 0 \\ -4 & 0 & -2 & 1 \\ 0 & 0 & 5 & 1 \\ 0 & 0 & 1 & 3 \end{pmatrix}$$

☐ [25%] Ch6(b): Let A be a 2×2 matrix satisfying Fourier's model

$$A\left(c_1\begin{pmatrix} 1 \\ -1 \end{pmatrix} + c_2\begin{pmatrix} -1 \\ 2 \end{pmatrix}\right) = 3c_1\begin{pmatrix} 1 \\ -1 \end{pmatrix} + 3c_2\begin{pmatrix} -1 \\ 2 \end{pmatrix}.$$

Prove or disprove: $\mathbf{x} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is an eigenvector of A.

☐ [50%] Ch6(c): Test the matrix A below to see if it is diagonalizable. If it is, then display the matrix packages P and D of eigenvectors and eigenvalues.

$$A = \begin{pmatrix} 5 & 1 & 1 \\ 0 & 5 & -1 \\ 0 & -1 & 5 \end{pmatrix}$$

(a) $4+\sqrt{2},\ 4-\sqrt{2},\ 4i,\ -4i$

(b) $\begin{pmatrix} 1 \\ 1 \end{pmatrix} = 3\begin{pmatrix} 1 \\ -1 \end{pmatrix} + 2\begin{pmatrix} -1 \\ 2 \end{pmatrix} =$ linear combination of eigenvectors for $\lambda = 3$

$\boxed{\text{yes}}$

(c) $P = \begin{pmatrix} 1 & -2 & 0 \\ 0 & 1 & -1 \\ 0 & 1 & 1 \end{pmatrix}$, $D = \begin{pmatrix} 5 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 6 \end{pmatrix}$

Staple this page to the top of all Ch6 work. Submit one package per chapter.

Name KEY

Differential Equations and Linear Algebra 2250-2

Final Exam 10:10am 13 December 2006

Ch7. (Linear Systems of Differential Equations)

☐ [50%] Ch7(a): Apply the eigenanalysis method to solve the system $\mathbf{x}' = A\mathbf{x}$, given

$$A = \begin{pmatrix} 4 & 1 & 1 \\ 0 & 6 & 0 \\ 2 & -1 & 5 \end{pmatrix}$$

☐ [25%] Ch7(b): Solve for $x(t)$, $y(t)$ in the system below.

$$\begin{aligned} x' &= x - 7y, \\ y' &= 7x + y. \end{aligned}$$

☐ [25%] Ch7(c): Consider a 2×2 system $\mathbf{x}' = A\mathbf{x}$. Assume A has complex eigenvalues $\lambda = \pm\sqrt{3}i$. Prove that $\lim_{t\to\infty} |\mathbf{x}(t)| = \infty$ is false for every possible solution $\mathbf{x}(t)$.

If you solved (a), (b), (c), then go on to Ch10. Otherwise, continue here. You may select either (a) or (d) but not both. Only 3 parts will be graded. The maximum score is 75 if you select (d).

☐ [25%] Ch7(d): Let $x(t)$ and $y(t)$ be the amounts of salt in brine tanks A and B, respectively. Assume fresh water enters A at rate $r = 10$ gallons/minute. Let A empty to B at rate r, and let B empty at rate r. Assume the model brine tank model

$$\begin{cases} x'(t) = -\dfrac{r}{50}x(t), \\[2mm] y'(t) = \dfrac{r}{50}x(t) - \dfrac{r}{100}y(t), \\[2mm] x(0) = 10, \quad y(0) = 15. \end{cases}$$

Find the maximum amount of salt ever in tank B.

(a) $\vec{x}(t) = c_1 e^{3t}\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} + c_2 e^{6t}\begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} + c_3 e^{6t}\begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$

(b) $x = (c_1 \cos wt + c_2 \sin wt)e^{t}$, $w = 7$, $y = \dfrac{x - x'}{7}$

(c) $\vec{x} = c_1 \cos\sqrt{3}t\, \vec{v}_1 + c_2 \sin\sqrt{3}t\, \vec{v}_2$. The terms are harmonic oscillations, hence $\vec{x}$ can't go to ∞.

(d) $t = 10\ln(40/35)$, from $2x(t) = y(t)$, where $x(t) = 10e^{-t/5}$, $y(t) = 35e^{-t/10} - 20e^{-t/5}$

Staple this page to the top of all Ch7 work. Submit one package per chapter.

Name KEY

Differential Equations and Linear Algebra 2250-2

Final Exam 10:10am 13 December 2006

Ch10. (Laplace Transform Methods)

It is assumed that you have memorized the basic 4-item Laplace integral table and know the 6 basic rules for Laplace integrals. No other tables or theory are required to solve the problems below. If you don't know a table entry, then leave the expression unevaluated for partial credit.

☐ [35%] Ch10(a): Apply Laplace's method to the system to find a formula for $\mathcal{L}(y(t))$. Find a 2×2 system for $\mathcal{L}(x)$, $\mathcal{L}(y)$ [20%]. Solve it **only** for $\mathcal{L}(y)$ [15%]. Do not solve for $x(t)$ or $y(t)$!

$$\begin{aligned} x'' &= -2y, \\ y'' &= 5x, \\ x(0) &= 0, \quad x'(0) = 1, \\ y(0) &= 0, \quad y'(0) = -1. \end{aligned}$$

☐ [35%] Ch10(b): Solve for $f(t)$, given

$$\mathcal{L}(f(t)) = \left(\frac{d}{ds}\left(\mathcal{L}(t^2\cos 2t)\right)\right)\Big|_{s \to (s+3)} + \frac{s^2+1}{(s+2)^3} - \mathcal{L}\left(\frac{d}{dt}\frac{d}{dt}\left(t^2 \sin 2t\right)\right)$$

☐ [30%] Ch10(c): Find $f(t)$ by partial fraction methods, given

$$\mathcal{L}(f(t)) = \frac{2-2s}{s^2+2s} + \frac{9s^2-27}{(s-1)^2(s+2)}.$$

If you solved (a), (b) and (c), then you have 100%. To select (d), unmark one of the previous three. Only three parts will be graded. If you did not solve (a) or (b), then the maximum score is 95.

☐ [30%] Ch10(d): Apply Laplace's method to find a formula for $\mathcal{L}(x(t))$. **Do not** solve for $x(t)$! Document steps by reference to tables and rules.

$$x^{iv} + 4x'' = e^{-t}(5 + 4t + 7\cos 3t), \quad x(0) = x'(0) = x''(0) = 0, \quad x'''(0) = -1.$$

(a) $\begin{pmatrix} s^2 & 2 \\ -5 & s^2 \end{pmatrix}\begin{pmatrix} \mathcal{L}(x) \\ \mathcal{L}(y) \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ $\quad \mathcal{L}(y) = \dfrac{5-s^2}{s^4+10}$

(b) $f(t) = -e^{-3t}t^3\cos 2t + (1 - 4t + 5t^2/2)e^{-2t} - (t^2 \sin 2t)''$

(c) $f(t) = 1 - 2e^{-2t} + 8e^t - 6te^t$

(d) $\mathcal{L}(x) = \dfrac{\text{top}}{\text{bot}}$, $\quad \text{top} = -1 + \dfrac{5}{s+1} + \dfrac{4}{(s+1)^2} + \dfrac{7(s+1)}{(s+1)^2+9}$

$\text{bot} = s^4 + 4s^2$

Staple this page to the top of all Ch10 work. Submit one package per chapter.