14 Dec 2006

Name KEY

Differential Equations and Linear Algebra 2250-1

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Ch3. (Linear Systems and Matrices)

☐ [25%] Ch3(a): Let B be the invertible matrix given below, where $\boxed{?}$ means the value of the entry does not affect the answer to this problem. The second matrix C is the adjugate (or adjoint) of B. Find the value of $\det(B)$.

$$B = \begin{pmatrix} ? & ? & ? & 0 \\ 1 & 1 & 0 & 0 \\ ? & ? & ? & 0 \\ ? & ? & ? & 3 \end{pmatrix}, \quad C = \begin{pmatrix} -6 & -6 & 0 & 0 \\ 6 & 6 & -6 & 0 \\ 3 & 0 & -3 & 0 \\ 2 & 2 & 0 & -2 \end{pmatrix}$$

☐ [25%] Ch3(b): State the three possibilities for a linear system $A\mathbf{x} = \mathbf{b}$. Determine which values of k correspond to these three possibilities, for the system $A\mathbf{x} = \mathbf{b}$ given in the display below.

$$A = \begin{pmatrix} 2 & 2 & -k \\ 2 & k & -2 \\ 0 & 0 & k \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$$

☐ [25%] Ch3(c): Assume A is an $n \times n$ matrix. A theorem says that $\det(A) \neq 0$ implies $A\mathbf{x} = \mathbf{b}$ has a unique solution. State two more linear algebra theorems, each with the same conclusion: $A\mathbf{x} = \mathbf{b}$ has a unique solution.

☐ [25%] Ch3(d): Find the value of x_1 by Cramer's Rule in the system $C\mathbf{x} = \mathbf{b}$, given C and $\mathbf{b}$ below. Evaluate determinants by any hybrid method (triangular, swap, combo, multiply, cofactor). The use of 2×2 Sarrus' rule is allowed (3×3 use is disallowed).

$$C = \begin{pmatrix} 1 & 1 & -1 & 0 \\ 1 & 3 & -2 & 1 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 2 & 1 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 0 \\ -1 \\ -1 \\ 1 \end{pmatrix}$$

If you solved (a), (b), (c) and (d), then go on to Ch4. Otherwise, continue here. Only four parts will be graded.

☐ [25%] Ch3(e): Give an example of a matrix A with 5 rows and 2 columns such that $A\mathbf{x} = \mathbf{0}$ has a unique solution.

(a) $BC = \det(B)I \Rightarrow \det(B) = (1\ 1\ 0\ 0)\begin{pmatrix} -6 \\ 6 \\ 0 \\ 2 \end{pmatrix} = \boxed{0}$

(b) 1. ∞-many sols, $k=0$. 2. No solution, $k \neq 0$ and $k=2$, 3. unique sol $\begin{cases} k \neq 0 \\ k-2 \neq 0 \end{cases}$

(c) 1. A^{-1} exists $\Rightarrow$ unique sol. 2. rref$(A) = I \Rightarrow$ unique sol. 3. Zero free variables $\Rightarrow$ unique sol.

(d) $x_1 = \frac{\Delta_1}{\Delta}$, $\Delta_1 = 9$, $\Delta = 8$

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Ch4. (Vector Spaces)

☐ [25%] Ch4(a): State (1) an RREF or rank test and (2) a determinant test to detect the independence or dependence of fixed vectors $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$ [10%]. Apply one of the tests to the vectors below [10%]. Report **independent** or **dependent** [5%].

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} -1 \\ 0 \\ 3 \end{pmatrix}, \quad \mathbf{v}_3 = \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}.$$

☐ [25%] Ch4(b): Let $\mathbf{v}_1$, $\mathbf{v}_2$ be a basis for a subspace S of vector space $V = \mathcal{R}^n$. Let $\mathbf{w}_1$, $\mathbf{w}_2$ be any other proposed basis of S. State a test which can decide whether or not the two bases are equivalent.

☐ [50%] Ch4(c): Define the 4×4 matrix A by the display below. Find a basis of fixed vectors in $\mathcal{R}^4$ for (1) the column space of A [25%] and (2) the row space A [25%]. The two displayed bases **must** consist of columns of A and rows of A, respectively.

$$A = \begin{pmatrix} 1 & -1 & 1 & 0 \\ 3 & -3 & -3 & -2 \\ 2 & -2 & -1 & -1 \\ 1 & -1 & -2 & -1 \end{pmatrix}$$

If you did (a), (b) and (c), then 100% has been marked – go on to Ch5. Otherwise, unmark one of (a), (b) or (c), then complete (d). Only three problems will be graded.

☐ [25%] Ch4(d): Define S to be the set of all vectors $\mathbf{x}$ in $\mathcal{R}^4$ such that $x_1 = x_3 + x_4$ and $x_1 - 2x_4 = 1$. Prove or disprove that S is a subspace of $\mathcal{R}^4$.

(a) $A = \text{aug}(v_1, v_2, v_3)$. Test I. $\{v_1, v_2, v_3\}$ independent $\Leftrightarrow$ rank$(A) = 3$

Test II, assume A is square. Then $\{v_1, v_2, v_3\}$ independent $\Leftrightarrow \det(A) \neq 0$.

By Test I, given vectors are dependent.

(b) $B = \text{aug}(v_1, v_2)$ $C = \text{aug}(w_1, w_2)$ $D = \text{aug}(v_1, v_2, w_1, w_2)$. Then bases are equivalent $\Leftrightarrow$ rank B = rank C = rank $D = 2$.

(c) pivots of A^T are 1, 2 and of A are 1, 3. Col$(A,1)$, Col$(A,3)$ = Basis of Col space

row$(A,1)$, row$(A,2)$ = Basis of row space

(d) Not a subspace, because $\vec{0}$ is not in S.

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Ch5. (Linear Equations of Higher Order)

☐ [10%] Ch5(a): Find the general solution of the differential equation

$$y'' + \frac{26}{5}y' + y = 0.$$

☐ [20%] Ch5(b): Find the homogeneous differential equation general solution, given characteristic equation

$$(r^2 - 5r)^3(r^4 - 25r^2)(r^2 + 2r + 5) = 0.$$

☐ [10%] Ch5(c): Given a damped spring-mass system $mx''(t) + cx'(t) + kx(t) = 0$ with $m = 15$, $c = 2$ and $k = 5$, classify the answer as over-damped, critically damped or under-damped. Please, **do not solve** the differential equation!

☐ [40%] Ch5(d): Assume a fifth order constant-coefficient differential equation has characteristic equation $r(r^2 + 4)(r^2 - 16)^2 = 0$. Suppose the right side of the differential equation is

$$f(x) = x(x + 2e^{4x}) + \sin 2x + 4x\cos 2x$$

Determine the **corrected** trial solution for y_p according to the method of undetermined coefficients. **Do not evaluate** the undetermined coefficients!

☐ [20%] Ch5(e): Find the steady-state periodic solution for the equation

$$x'' + 4x' + 5x = \cos(2t).$$

(a) $y = c_1 e^{-5x} + c_2 e^{-x/5}$

(b) roots $= 0,0,0,0,0,\ 5,5,5,5,\ -5,\ -1+2i, -1-2i$

$L = \{1, x, x^2, x^3, x^4,\ e^{5x}, xe^{5x}, x^2e^{5x}, x^3e^{5x},\ e^{-5x},\ e^{-x}\cos 2x, e^{-x}\sin 2x\}$

$y =$ linear combination of the atoms in List L.

(c) Discriminant $< 0 \Rightarrow$ under damped

(d) $y = (d_1\cos 2x + d_2\sin 2x + d_3 x\cos 2x + d_4 x\sin 2x)x$
$\quad + (d_5 + d_6 x + d_7 x^2)x$
$\quad + (d_8 e^{4x} + d_9 xe^{4x})x^2$

(e) $x_{ss} = \frac{1}{65}\cos(2t) + \frac{8}{65}\sin(2t)$

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Ch6. (Eigenvalues and Eigenvectors)

☐ [25%] Ch6(a): Find the eigenvalues of the matrix

$$A = \begin{pmatrix} 0 & 4 & -1 & 0 \\ 4 & 0 & -2 & 1 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 1 & 4 \end{pmatrix}.$$

☐ [25%] Ch6(b): Let A be a 3×3 matrix satisfying Fourier's model

$$A\left(c_1\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} + c_2\begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix} + c_3\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}\right) = 3c_1\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} + 3c_2\begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix} + 2c_3\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

Prove or disprove: $\mathbf{x} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ is an eigenvector of A.

☐ [50%] Ch6(d): The matrix A below has eigenvalue package

$$D = \begin{pmatrix} 6 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

Test A to see it is diagonalizable, and if it is, then display the matrix package P of eigenvectors.

$$A = \begin{pmatrix} 5 & 1 & 1 \\ 1 & 5 & -1 \\ 1 & -1 & 5 \end{pmatrix}$$

(a) $\lambda = 4, -4, 3+\sqrt{2}, 3-\sqrt{2}$

(b) $\begin{pmatrix} 1 & -1 & 1 \\ -1 & 2 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ has rank 2. Then Col 3 depends on Col 1, 2. So $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ is an eigenvector for $\lambda = 3$, because $\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix}$ are eigenvectors for $\lambda = 3$.

$\boxed{\text{yes, an eigenvector}}$

(c) yes, $P = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$, $D = \begin{bmatrix} 6 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

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Ch7. (Linear Systems of Differential Equations)

☐ [50%] Ch7(a): Apply the eigenanalysis method to solve the system $\mathbf{x}' = A\mathbf{x}$, given

$$A = \begin{pmatrix} 5 & -1 & -1 \\ 0 & 3 & 0 \\ -2 & 1 & 4 \end{pmatrix}$$

☐ [25%] Ch7(b): Give an example of a 2×2 real matrix A for which Fourier's model is not valid. Then display the general solution $\mathbf{x}(t)$ of $\mathbf{x}' = A\mathbf{x}$.

☐ [25%] Ch7(c): Solve for $y(t)$ in the system below.

$$\begin{aligned} x' &= x + 3y, \\ y' &= -3x + y. \end{aligned}$$

If you solved (a), (b), (c), then you have marked 100%. If so, then go on to Ch10, otherwise, continue here. You may select either (a) or (d) but not both. Only 3 parts will be graded. The maximum score is 75 if you select (d).

☐ [25%] Ch7(d): Let $x(t)$ and $y(t)$ be the amounts of salt in brine tanks A and B, respectively. Assume fresh water enters A at rate $r = 10$ gallons/minute. Let A empty to B at rate r, and let B empty at rate r. Assume the model brine tank model

$$\begin{cases} x'(t) = -\dfrac{r}{100}x(t), \\ y'(t) = \dfrac{r}{100}x(t) - \dfrac{r}{50}y(t), \\ x(0) = 5, \quad y(0) = 10. \end{cases}$$

Find the maximum amount of salt ever in tank B.

(a) $\vec{x}(t) = c_1 e^{6t}\begin{pmatrix}-1\\0\\1\end{pmatrix} + c_2 e^{3t}\begin{pmatrix}0\\-1\\1\end{pmatrix} + c_3 e^{3t}\begin{pmatrix}1\\2\\0\end{pmatrix}$ another eigenpair is $\left(3, \begin{pmatrix}1\\1\\1\end{pmatrix}\right)$ which appeared often.

(b) $A = \begin{pmatrix}0 & 1\\0 & 0\end{pmatrix}$ has only one eigenvector $\begin{pmatrix}1\\0\end{pmatrix}$ for $\lambda = 0$. Then $\begin{cases} x = c_1 + c_2 t \\ y = c_2 \end{cases}$ is the sol.

(c) $y(t) = c_1 e^{t} \cos 3t + c_2 e^{t} \sin 3t$

(d) $x(t) = 5e^{-t/10}$, $y(t) = 5e^{-t/10} + 5e^{-t/5}$ $\quad \max_{t \geq 0} y(t) = y(0) = 10$

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Ch10. (Laplace Transform Methods)

It is assumed that you have memorized the basic 4-item Laplace integral table and know the 6 basic rules for Laplace integrals. No other tables or theory are required to solve the problems below. If you don't know a table entry, then leave the expression unevaluated for partial credit.

☐ [35%] Ch10(a): Apply Laplace's method to the system to find a formula for $\mathcal{L}(y(t))$. Find a 2×2 system for $\mathcal{L}(x)$, $\mathcal{L}(y)$ [20%]. Solve it **only** for $\mathcal{L}(y)$ [15%]. Do not solve for $x(t)$ or $y(t)$!

$$\begin{aligned} x'' &= -3y, \\ y'' &= 4x, \\ x(0) &= 0, \quad x'(0) = 1, \\ y(0) &= 0, \quad y'(0) = 0. \end{aligned}$$

☐ [35%] Ch10(b): Solve for $f(t)$, given

$$\mathcal{L}(f(t)) = \left(\frac{d}{ds}\left(\mathcal{L}(t^2\cos 2t)\right)\right)\bigg|_{s\to(s+3)} + \frac{s^2+2}{(s+1)^3} + \mathcal{L}\left(\frac{d}{dt}\left(e^t \sin 2t\right)\right)$$

☐ [30%] Ch10(c): Find $f(t)$ by partial fraction methods, given

$$\mathcal{L}(f(t)) = \frac{2+2s}{s^2-2s} + \frac{9s^2-27}{(s+1)^2(s+2)}.$$

If you solved (a), (b) and (c), then you have 100%. To select (d), unmark one of the previous three. Only three parts will be graded. If you did not solve (a) or (b), then the maximum score is 95.

☐ [30%] Ch10(d): Apply Laplace's method to find a formula for $\mathcal{L}(x(t))$. **Do not** solve for $x(t)$! Document steps by reference to tables and rules.

$$x^{iv} + 4x'' = e^t(5t + 4e^t + 3\sin 3t), \quad x(0) = x'(0) = x''(0) = 0. \quad x'''(0) = -1.$$

(a) $\mathcal{L}(y) = \frac{\Delta_2}{\Delta}$, $\Delta = \begin{vmatrix} s^2 & 3 \\ -4 & s^2 \end{vmatrix}$, $\Delta_2 = \begin{vmatrix} s^2 & 1 \\ -4 & 0 \end{vmatrix}$. $\mathcal{L}(y) = \frac{4}{s^4+12}$

(b) $f(t) = -t^3 e^{-3t}\cos 2t + (1 - 2t + 3t^2/2)e^{-t} + (e^t \sin 2t)'$

(c) $f(t) = -1 + 3e^{2t} + 9e^{-2t} - 18te^{-t}$

(d) $\mathcal{L}(x) = \frac{\text{top}}{\text{bot}}$, $\text{top} = -1 + \frac{5}{(s-1)^2} + \frac{4}{s-2} + \frac{9}{(s-1)^2+9}$, $\text{bot} = s^4 + 4s^2$

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