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Differential Equations and Linear Algebra 2250-1

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Ch3. (Linear Systems and Matrices)

☐ [50%] Ch3(a): Find the **fourth entry on the third row** of the inverse matrix B^{-1} by the formula $B^{-1} = \mathbf{adj}(B)/\det(B)$. Evaluate determinants by any method: triangular, swap, combo, multiply, cofactor. The use of the 2×2 Sarrus' rule is expected.

$$B = \begin{bmatrix} 1 & 1 & -2 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 2 & 0 & 0 \\ 1 & 0 & 0 & 3 & 0 \\ -1 & 0 & 1 & 3 & 4 \end{bmatrix}$$

☐ [25%] Ch3(b): Determine all values of k such that the system $R\mathbf{x} = \mathbf{f}$ has no solution.

$$R = \begin{bmatrix} 2 & 1 & k \\ 2 & -k & -2 \\ 0 & 0 & k \end{bmatrix}, \quad \mathbf{f} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

☐ [25%] Ch3(c): Let A be a 3×3 triangular matrix with diagonal entries 7, -11, 1. Prove that $A\mathbf{x} = \mathbf{0}$ has only the solution $\mathbf{x} = \mathbf{0}$.

☐ [25%] Ch3(d): Let A denote a 3×4 matrix. Explain from theory why $A\mathbf{x} = \mathbf{0}$ has infinitely many solutions.

☐ [25%] Ch3(e): Infinitely many 3×3 matrices A exist such that A^3 is the zero matrix but A^2 is not the zero matrix. Display one such matrix A and justify the claim.

Ch3(a) $\text{Cof}(B,4,3) = (-1)^{4+3}\begin{vmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 3 & 4 \end{vmatrix} = 0.$ B^{-1} entry $= \dfrac{\text{cofactor}}{\text{determinant}} = \boxed{0}$

$\det(B) = 24$

Ch3(b) Sequence to rref stops at $\left(\begin{array}{ccc|c} 2 & 1 & k & 0 \\ 0 & -k-1 & -2-k & 1 \\ 0 & 0 & k & 0 \end{array}\right)$. If $-k-1=0$, Then $\left(\begin{array}{ccc|c} 2 & 1 & -1 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & -1 & 0 \end{array}\right)$ give a signal eq after combo. Otherwise, a sol always exists.

answer: $\boxed{\text{No sol } k=-1}$

Ch3(c) Then $\det(A) = 7(-11)(1) = -77 \neq 0$, So A^{-1} exists and $\vec{x} = A^{-1}A\vec{x} = A^{-1}\vec{0} = \vec{0}$

Ch3(d) rank + nullity = 4, Only 3 rows $\Rightarrow$ rank ≤ 3 $\Rightarrow$ nullity ≥ 1 $\Rightarrow$ one free var. Therefore, at least one free var implies ∞-many solutions.

Ch3(e) $A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$. Then $A^2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $A^3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

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Ch4. (Vector Spaces)

☐ [40%] Ch4(a): State an RREF test (not a determinant test) to detect the independence or dependence of fixed vectors $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}3$ in $\mathcal{R}^4$ [10%]. Apply the test to the vectors below [25%]. Report **independent** or **dependent** [5%].

$$\mathbf{v}_1 = \begin{pmatrix} -1 \\ 1 \\ 2 \\ 1 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} 3 \\ 0 \\ 1 \\ 1 \end{pmatrix}, \quad \mathbf{v}_3 = \begin{pmatrix} 4 \\ -1 \\ -1 \\ 0 \end{pmatrix}.$$

☐ [60%] Ch4(b): Define V to be the set of all vectors $\mathbf{x}$ in $\mathcal{R}^4$ such that $x_1 + x_4 = 0$ and $\mathbf{c} \cdot \mathbf{x} = 0$, where $\mathbf{c} = \begin{pmatrix} -1 \\ 0 \\ 1 \\ 3 \end{pmatrix}$. Prove that V is a subspace of $\mathcal{R}^4$.

☐ [60%] Ch4(c): Find a basis of fixed vectors in $\mathcal{R}^4$ for (1) the column space of the 4×4 matrix A below [30%] and (2) the row space of the 4×4 matrix A below [30%]. The two displayed bases **must** consist of columns of A and rows of A, respectively.

$$A = \begin{pmatrix} 1 & -1 & 1 & 0 \\ 1 & -1 & -2 & -1 \\ 2 & -2 & -1 & -1 \\ 3 & -3 & 0 & -1 \end{pmatrix}$$

☐ [40%] Ch4(d): Find a 4×4 system of linear equations for the constants a, b, c, d in the partial fractions decomposition below [10%]. Solve for a, b, c, d, showing all **RREF** steps [25%]. Report the answers [5%].

$$\frac{4x^2 - 12x + 4}{(x-1)^2(x+1)^2} = \frac{a}{x-1} + \frac{b}{(x-1)^2} + \frac{c}{x+1} + \frac{d}{(x+1)^2}$$

Ch4 (a) Test: independent $\iff$ rank(aug(v_1, v_2, v_3)) $= 3$. Apply: $\boxed{\text{independent.}}$

Ch4 (b) $A = \begin{pmatrix} 1 & 0 & 0 & 1 \\ -1 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$. The restriction eqs are $A\vec{x} = \vec{0}$. Apply Theorem 2 of E&P.

Ch4 (c) Stop in rref sequence at $\begin{pmatrix} 1 & -1 & 1 & 0 \\ 0 & 0 & -3 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$. Then col 1, 3 = pivots = indep cols of A. Repeat with A^T to get rows 1, 2 = indep rows of A

Ch4 (d) Heaviside coverup gives $\boxed{b=-1}$ and $\boxed{d=5}$. Clear fractions, then substitute $x=0, x=2$ to get a 2x2 system for a, c. Solve it. $\boxed{a=0}$ $\boxed{c=0}$

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Ch5. (Linear Equations of Higher Order)

☐ [25%] Ch5(a): Using the *recipe* for higher order constant-coefficient differential equations, write out the general solutions of the differential equations whose characteristic equations are given below.

$$\mathbf{1}.[12\%]\quad r^4(r^2-5r)^2(r^2-25)=0,$$
$$\mathbf{2}.[13\%]\quad (r+4)^2(r^2+2r+2)^3(r^2-16)^2=0$$

☐ [25%] Ch5(b): Given a damped spring-mass system $mx''(t)+cx'(t)+kx(t)=0$ with $m=15$, $c=17$ and $k=4$, solve the differential equation [25%] and classify the answer as over-damped, critically damped or under-damped [5%].

☐ [50%] Ch5(c): Determine the **corrected** trial solution for y_p according to the method of undetermined coefficients. **Do not evaluate** the undetermined coefficients!

$$y^{iv}+4y''=x^2(1+2e^{2x})+3x\sin 2x+2\sin x\cos x$$

☐ [25%] Ch5(d): Find the steady-state periodic solution for the equation

$$x''+4x'+20x=3\cos(2t).$$

Ch5(a) 1. roots = 0,0,0,0,0,0, 5,5,5,−5. $y = u_1e^{0x}+u_2e^{5x}+u_3e^{-5x}$,
$u_1 = c_1+c_2x+c_3x^2+c_4x^3+c_5x^4+c_6x^5$, $u_2=c_7+c_8x+c_9x^2$, $u_3=c_{10}$

2. roots = −4,−4,−4,−4, 4,4, $-1\pm i$, $-1\pm i$, $-1\pm i$ (12 roots)
$y=u_1e^{-4x}+u_2e^{4x}+u_3e^{-x}\cos x+u_4e^{-x}\sin x$, $u_1=c_1+c_2x+c_3x^2+c_4x^3$
$u_2=c_5+c_6x$, $u_3=c_7+c_8x+c_9x^2$, $u_4=c_{10}+c_{11}x+c_{12}x^2$

Ch5(b) $(5r+4)(3r+1)=0$ has roots $-4/5, -1/3$. overdamped. Sol is
$x(t)=c_1e^{-4t/5}+c_2e^{-t/3}$

Ch5(c) $r^2(r^2+4)=0$ homogeneous roots $=0,0,\pm 2i$. Since $2\sin x\cos x=\sin 2x$,
the corrected trial sol is $y=y_1+y_2+y_3$, where $y_1=x^2(d_1+d_2x+d_3x^2)$,
$y_2=(d_4+d_5x+d_6x^2)e^{2x}$, $y_3=x([d_7+d_8x]\cos 2x+[d_9+d_{10}x]\sin 2x)$

Ch5(d) Trial Sol $x=d_1\cos 2t+d_2\sin 2t$ gives eqs $\begin{cases}16d_1+8d_2=3\\ -8d_1+16d_2=0\end{cases}$
Cramer's rule
$\Delta=320$
$\Delta_1=(16)(3)$
$\Delta_2=(8)(3)$
Solving, $d_1=3/20$, $d_2=3/40$

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Ch6. (Eigenvalues and Eigenvectors)

☐ [25%] Ch6(a): Find the eigenvalues of the matrix A:

$$A = \begin{bmatrix} 3 & 1 & -1 & 0 \\ 0 & 3 & -2 & 1 \\ 0 & 0 & 5 & 1 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

☐ [25%] Ch6(b): Let A be a 2×2 matrix with eigenpairs

$$(\lambda_1, \mathbf{v}_1), \quad (\lambda_2, \mathbf{v}_2).$$

Display Fourier's model for the matrix A.

☐ [25%] Ch6(c): Assume two 3×3 matrices A, B have exactly the same characteristic equations. Let A have eigenvalues 2, 3, 4. Find the eigenvalues of $(1/3)B - 2I$, where I is the identity matrix.

☐ [25%] Ch6(d): Let 3×3 matrices A and B be related by $AP = PB$ for some invertible matrix P. Prove that the roots of the characteristic equations of A and B are identical.

☐ [25%] Ch6(e): Let A be a 3×3 matrix with eigenpairs

$$(5, \mathbf{v}_1), \quad (3, \mathbf{v}_2), \quad (-1, \mathbf{v}_3).$$

Let $P = \mathbf{aug}(\mathbf{v}_2, \mathbf{v}_1, \mathbf{v}_3)$. Display the answer for $P^{-1}AP$ [20%]. Justify your claim with a sentence [5%].

Ch6ⓐ $(3-\lambda)(3-\lambda)(\lambda^2-8\lambda+14)=0 \qquad \lambda = 3, 3, 4+\sqrt{2}, 4-\sqrt{2}$

Ch6ⓑ $\vec{x} = c_1\vec{v}_1 + c_2\vec{v}_2 \Rightarrow A\vec{x} = c_1\lambda_1\vec{v}_1 + c_2\lambda_2\vec{v}_2$

Ch6ⓒ $\det\left(\frac{1}{3}B - 2I - \lambda I\right) = \det\left(\frac{1}{3}I\right)\det(B - 6I - 3\lambda I)$

$0 = \det(B - (6+3\lambda)I)$ Find eigenvalues λ of B

$0 = \det(A - (6+3\lambda)I)$ A, B have same char. eq.

Then $6+3\lambda$ = eigenvalue of A $\Rightarrow \lambda = \frac{1}{3}(2-6), \frac{1}{3}(3-6), \frac{1}{3}(4-6) \Rightarrow \lambda = \frac{-4}{3}, -1, \frac{-2}{3}$

Ch6ⓓ $\det(A - \lambda I) = \det(PBP^{-1} - \lambda PIP^{-1})$

$= \det P \det(B - \lambda I) \det P^{-1}$

$= \det(B - \lambda I)$ because $\det P \det P^{-1} = \det PP^{-1} = 1$

Ch6ⓔ $D = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & -1 \end{pmatrix}$. Then $PD = \text{aug}(3v_2, 5v_1, -v_3) = AP$.

$3\lambda + 6 = \lambda_A$

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Ch7. (Linear Systems of Differential Equations)

☐ [50%] Ch7(a): Apply the eigenanalysis method to solve the system $\mathbf{x}' = A\mathbf{x}$, given

$$A = \begin{bmatrix} -3 & 1 & 1 \\ 1 & -3 & 1 \\ 0 & 0 & -4 \end{bmatrix}$$

☐ [25%] Ch7(b): Solve for $y(t)$ in the system below. Don't solve for $x(t)$!

$$\begin{aligned} x' &= x + 2y, \\ y' &= -2x + y. \end{aligned}$$

☐ [25%] Ch7(c): Let A be a 4×4 real matrix and assume Fourier's model is valid for A:

$$\mathbf{x} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 + c_4\mathbf{v}_4 \quad \text{implies} \quad A\mathbf{x} = c_1\lambda_1\mathbf{v}_1 + c_2\lambda 2\mathbf{v}_2 + c_3\lambda_3\mathbf{v}_3 + c_4\lambda_4\mathbf{v}_4$$

Display the general solution $\mathbf{x}(t)$ for $\mathbf{x}' = A\mathbf{x}$ in terms of the ingredients of Fourier's model.

☐ [25%] Ch7(d): Consider a 3×3 system $\mathbf{x}' = A\mathbf{x}$. Assume A has an eigenvalue $\lambda = -0.1$ with corresponding eigenvectors

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}.$$

Find a nonzero solution of the differential equation with limit zero at infinity.

☐ [25%] Ch7(e): Assume A is 2×2 and has eigenvalues $2 \pm \sqrt{7}i$. In the system $\mathbf{u}' = A\mathbf{u}$ where $\mathbf{u}(t)$ has components $x(t)$, $y(t)$, explain why

$$y(t) = c_1 e^{2t}\cos\sqrt{7}\,t + c_2 e^{2t}\sin\sqrt{7}\,t.$$

☐ [25%] Ch7(f): Let $x(t)$ and $y(t)$ be the amounts of salt in brine tanks A and B, respectively. Assume fresh water enters A at rate $r = 5$ gallons/minute. Let A empty to B at rate r, and let B empty at rate r. Assume the model

$$\begin{cases} x'(t) = -\dfrac{r}{50}x(t), \\[2mm] y'(t) = \dfrac{r}{50}x(t) - \dfrac{r}{100}y(t), \\[2mm] x(0) = 10, \quad y(0) = 5. \end{cases}$$

Find the maximum amount of salt ever in tank B.

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(a) $\lambda_1 = \lambda_2 = -4, \quad \lambda_3 = -2 \qquad v_1 = \begin{bmatrix} -1 \\ -1 \\ 0 \end{bmatrix}, \; v_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \; v_3 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$

$$x(t) = c_1 e^{-4t} v_1 + c_2 e^{-4t} v_2 + c_3 e^{-2t} v_3$$

(b) $y'' - 2y' + 5y = 0 \qquad r^2 - 2r + 5 = 0 \qquad r_{1,2} = 1 \pm 2i$

$$y(t) = e^t (c_1 \cos(2t) + c_2 \sin(2t)).$$

(c) $$x(t) = c_1 e^{-\lambda_1 t} v_1 + c_2 e^{\lambda_2 t} v_2 + c_3 e^{\lambda_3 t} v_3 + c_4 e^{\lambda_4 t} v_4$$

(d) Any non-trivial linear combination of $\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} e^{-0.1t}$ and $\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} e^{-0.1t}$

Since $e^{-0.1t} \to 0$ when $t \to \infty$, solutions $\to 0$.

(e) Both $x(t)$, $y(t)$ are solutions of the corresponding second-order equation whose characteristic equation has $2 \pm \sqrt{7}\, i$ as roots. Hence the formula for y.

(f) $$X(t) = 10 \cdot e^{-t/10}$$

$$y(t) = 25 e^{-t/20} - 20 e^{-t/10}$$

Max y at $y'(t) = 0 \Rightarrow y(t) = 2(x(t))$

$\Rightarrow 25 e^{-t/20} = 40 e^{-t/10} \qquad t = 20 \ln(8/5)$

Evaluate y at $20 \ln(8/5)$

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Ch10. (Laplace Transform Methods)

It is assumed that you have memorized the basic 4-item Laplace integral table and know the 6 basic rules for Laplace integrals. No other tables or theory are required to solve the problems below. If you don't know a table entry, then leave the expression unevaluated for partial credit.

☐ [35%] Ch10(a): Apply Laplace's method to the system to find a formula for $\mathcal{L}(y(t))$. Find a 2×2 system for $\mathcal{L}(x)$, $\mathcal{L}(y)$ [20%]. Solve it **only** for $\mathcal{L}(y)$ [15%]. Do not solve for $x(t)$ or $y(t)$!

$$\begin{aligned} x'' &= 4x - y, \\ y'' &= 2x + 2y, \\ x(0) &= 0, \quad x'(0) = 3, \\ y(0) &= 0, \quad y'(0) = 4. \end{aligned}$$

☐ [35%] Ch10(b): Solve for $x(t)$, given

$$\mathcal{L}(x(t)) = \frac{d}{ds}\left(\mathcal{L}(e^t \sin 2t)\right) + \frac{s^2}{(s-1)^3} + \frac{1+s}{s^2-5s} - \left(\frac{s-1}{s^2+1}\right)\Big|_{s \to (s-2)}.$$

☐ [30%] Ch10(c): Find $f(t)$ by partial fraction methods, given

$$\mathcal{L}(f(t)) = \frac{8s^2 - 24}{(s+1)(s-3)(s-1)^2}.$$

☐ [30%] Ch10(d): Apply Laplace's method to find a formula for $\mathcal{L}(x(t))$. **Do not** solve for $x(t)$! Document steps by reference to tables and rules.

$$x^{iv} + 4x'' = 3t^2 + 4te^{-2t} + e^t \sin 3t, \quad x(0) = x'(0) = x''(0) = 0, \quad x'''(0) = -1.$$

(a) $\begin{cases} (s^2-4)\mathcal{L}(x) + \mathcal{L}(y) = 3 \\ -2\,\mathcal{L}(x) + (s^2-2)\mathcal{L}(y) = 4 \end{cases}$ $\qquad \mathcal{L}(y) = \dfrac{\begin{vmatrix} s^2-4 & 3 \\ -2 & 4 \end{vmatrix}}{\begin{vmatrix} s^2-4 & 1 \\ -2 & s^2-2 \end{vmatrix}}$

(b) $x = x_1 + x_2 + x_3 + x_4$

$x_1 = -te^t \sin 2t$

$x_2 = e^t + 2te^t + \frac{1}{2}t^2e^t$

$x_3 = \frac{6}{5}e^{5t} - \frac{1}{5}$

$\dfrac{s^2}{(s-1)^3} = \dfrac{1}{s-1} + \dfrac{2}{(s-1)^2} + \dfrac{1}{(s-1)^3}$

$\dfrac{1+s}{s^2-5s} = \dfrac{6}{5}\cdot\dfrac{1}{s-5} - \dfrac{1}{5}\cdot\dfrac{1}{s}$

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$x_4 = e^{2t}(\cos(t) - \sin(t)).$

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(c)

$$\frac{8s^2-24}{(s+1)(s-3)(s-1)^2} = -\frac{4}{s-1} + \frac{4}{(s-1)^2} + \frac{1}{s+1} + \frac{3}{s+3}$$

$$f(t) = -4e^{t} + 4te^{t} + e^{-t} + 3e^{-3t}$$

(d)

$$s^4\mathcal{L}(x) + 1 + 4s^2\mathcal{L}(x) = \frac{6}{s^3} + \frac{4}{(s+2)^2} + \frac{3}{(s-1)^2+9}$$

$$\mathcal{L}(x) = \frac{1}{s^4+4s^2}\left[\frac{6}{s^3} + \frac{4}{(s+2)^2} + \frac{3}{(s-1)^2+9} - 1\right].$$