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Differential Equations and Linear Algebra 2250-2

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Ch3. (Linear Systems and Matrices)

☐ [50%] Ch3(a): Find the **third entry on the fifth row** of the inverse matrix B^{-1} by the formula $B^{-1} = \mathbf{adj}(B)/\det(B)$. Evaluate determinants by any method: triangular, swap, combo, multiply, cofactor. The use of 3×3 Sarrus' rule is disallowed (2×2 use is allowed).

$$B = \begin{bmatrix} 1 & 1 & -2 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 2 & 0 & 0 \\ 1 & 0 & 0 & 3 & 0 \\ -1 & 0 & 1 & 3 & 4 \end{bmatrix}$$

☐ [50%] Ch3(b): Determine all values of k such that the system $R\mathbf{x} = \mathbf{f}$ has (1) infinitely many solutions [15%], (2) a unique solution [15%] and (3) no solution [20%].

$$R = \begin{bmatrix} 2 & 1 & k \\ 2 & k & -2 \\ 0 & 0 & 4 \end{bmatrix}, \quad \mathbf{f} = \begin{pmatrix} 0 \\ 1+k \\ 0 \end{pmatrix}$$

☐ [25%] Ch3(c): Let A be a 3×3 triangular matrix with diagonal entries 3, 1, 0. Prove that $A\mathbf{x} = \mathbf{0}$ has infinitely many solutions $\mathbf{x}$.

☐ [25%] Ch3(d): Let A denote a 4×3 nonzero matrix. Find an example A such that $A\mathbf{x} = \mathbf{0}$ has a unique solution.

☐ [25%] Ch3(e): There are real 2×2 matrices A such that $A^2 = -I$, where I is the identity matrix. Display one such matrix A and justify the claim.

Ch3 ⓐ $\mathrm{Cof}(B,3,5) = (-1)^8 \begin{vmatrix} 1 & 1 & -2 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 3 \\ -1 & 0 & 1 & 3 \end{vmatrix} = 12$, $\det(B) = 24$ $\boxed{1/2}$

ch 3 ⓑ $\mathrm{aug}(R,f) \cong \left(\begin{array}{ccc|c} 2 & 1 & 0 & 0 \\ 0 & k-1 & 0 & 1+k \\ 0 & 0 & 1 & 0 \end{array}\right)$ (1) Never ∞-many (2) $k-1 \neq 0$ unique sol (3) $k=1$ signal eq no sol.

ch3 ⓒ Because $\det(A)=0$, Then ∞-many sol

ch3 ⓓ $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ unique sol $\vec{x} = \vec{0}$ to $A\vec{x} = \vec{0}$.

ch3 ⓔ Let $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, Then $A^2 = -I$

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Ch4. (Vector Spaces)

☐ [40%] Ch4(a): State an RREF test (not a determinant test) to detect the independence or dependence of fixed vectors $\mathbf{v}_1$, $\mathbf{v}_2$, v3 in $\mathcal{R}^5$ [10%]. Apply the test to the vectors below [25%]. Report **independent** or **dependent** [5%].

$$\mathbf{v}_1 = \begin{pmatrix} -1 \\ 1 \\ 2 \\ 0 \\ 1 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} 3 \\ 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}, \quad \mathbf{v}_3 = \begin{pmatrix} 4 \\ -1 \\ -1 \\ 0 \\ 1 \end{pmatrix}.$$

☐ [60%] Ch4(b): Define V to be the set of all vectors $\mathbf{x}$ in $\mathcal{R}^5$ such that $2x_1 + x_5 = x_3$ and $\mathbf{c} \cdot \mathbf{x} = 0$, where $\mathbf{c}$ has constant components c_1 through c_5. Prove that V is a subspace of $\mathcal{R}^5$.

☐ [60%] Ch4(c): Find a basis of fixed vectors in $\mathcal{R}^4$ for (1) the column space of the 4×4 matrix A below [30%] and (2) the row space of the 4×4 matrix A below [30%]. The reported basis must consist of columns of A and rows of A, respectively.

$$A = \begin{pmatrix} 1 & 1 & -1 & 0 \\ 1 & 1 & -1 & -4 \\ 0 & 0 & 0 & 5 \\ 2 & 2 & -2 & 6 \end{pmatrix}.$$

☐ [40%] Ch4(d): Find a 4×4 system of linear equations for the constants a, b, c, d in the partial fractions decomposition below [10%]. Solve for a, b, c, d, showing all **RREF** steps [25%]. Report the answers [5%].

$$\frac{3x^2 - 14x + 3}{(x+1)^2(x-2)^2} = \frac{a}{x+1} + \frac{b}{(x+1)^2} + \frac{c}{x-2} + \frac{d}{(x-2)^2}$$

Ch4(a) Test: v_1, v_2, v_3 independent in $\mathbb{R}^5 \Leftrightarrow \text{rank}(\text{aug}(v_1, v_2, v_3)) = 3$.
Apply: $\text{aug}(v_1, v_2, v_3)$ has 3 leading ones in rref $\Rightarrow$ rank $= 3 \Rightarrow$ $\boxed{\text{indep}}$

ch4(b) Let A have rows $(2, 0, -1, 0, 1), \vec{c}, \vec{0}, \vec{0}, \vec{0}$. Then the restriction eqs can be written as $A\vec{x} = \vec{0}$. Apply Theorem 2 E&P. Then V is a subspace.

ch4(c) Find rref(A), rref(AT). By the pivot theorem, $\boxed{\begin{array}{l} 1,4 = \text{independent cols of } A \\ 1,2 = \text{independent rows of } A \end{array}}$

ch4(d) $a = -20/27$, $b = 20/9$, $c = 20/27$, $d = -13/9$

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Ch5. (Linear Equations of Higher Order)

☐ [25%] Ch5(a): Using the *recipe* for higher order constant-coefficient differential equations, write out the general solutions of the differential equations whose characteristic equations are given below.

1.[1~~6~~%] 12 $\quad r^3(r^2-5r)^2(r^2-25)=0,$

2.[1~~5~~%] 13 $\quad (r-4)^2(r^2+2r+3)^2(r^2-16)^3=0$

☐ [25%] Ch5(b): Given a damped spring-mass system $mx''(t)+cx'(t)+kx(t)=0$ with $m=10$, $c=13$ and $k=4$, solve the differential equation [25%] and classify the answer as over-damped, critically damped or under-damped [5%].

☐ [50%] Ch5(c): Determine the **corrected** trial solution for y_p according to the method of undetermined coefficients. **Do not evaluate** the undetermined coefficients!

$$y^{iv}+9y''=x(x+2e^{3x})+3x\cos 3x+4e^{-3x}$$

☐ [25%] Ch5(d): Find the steady-state periodic solution for the equation

$$x''+4x'+29x=\cos(3t).$$

Ch5(a) 1. roots = 0,0,0,0,0,5,5,5,−5. $y=u_1e^{0x}+u_2e^{5x}+u_3e^{-5x}$, $u_1=c_1+c_2x+c_3x^2+c_4x^3+c_5x^4$, $u_2=c_6+c_7x+c_8x^2$, $u_3=c_9$

2. roots = 4,4, $-1\pm\sqrt2 i$, $-1\pm\sqrt2 i$, 4,4,4,−4,−4,−4 (12 roots), $y=u_1e^{4x}+u_2e^{-x}\cos\sqrt2x+u_3e^{-x}\sin\sqrt2x+u_4e^{-4x}$, $u_1=c_1+c_2x+c_3x^2+c_4x^3+c_5x^4$, $u_2=c_6+c_7x$, $u_3=c_8+c_9x$, $u_4=c_{10}+c_{11}x+c_{12}x^2$

Ch5(b) overdamped, roots = $-4/5, -1/2$, $x(t)=c_1e^{-4t/5}+c_2e^{-t/2}$

Ch5(c) $y=y_1+y_2+y_3+y_4$, $y_1=(d_1+d_2x+d_3x^2)x^2$, $y_2=(d_4+d_5x)e^{3x}$, $y_3=[(d_6+d_7x)\cos 3x+(d_8+d_9x)\sin 3x]x$, $y_4=d_{10}e^{-3x}$

Fixup rule was applied to y_1, y_3 only.

Ch5(d) Trial sol $x=d_1\cos 3t+d_2\sin 3t$ is the unique periodic solution.

$d_1=\frac{20}{\Delta}$, $d_2=\frac{12}{\Delta}$, $\Delta=400+144=544$ found from the equations

$\begin{cases}20d_1+12d_2=1\\-12d_1+20d_2=0\end{cases}$ Simplified, $d_1=\frac{5}{136}$, $d_2=\frac{3}{136}$.

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Ch6. (Eigenvalues and Eigenvectors)

☐ [~~30~~ 25%] Ch6(a): Find the eigenvalues of the matrix A:

$$A = \begin{bmatrix} 2 & 4 & -1 & 0 \\ 0 & 5 & -2 & 1 \\ 0 & 0 & 4 & 1 \\ 0 & 0 & 1 & 4 \end{bmatrix}$$

☐ [25%] Ch6(b): Let A be a 3×3 matrix with eigenpairs

$$(5, \mathbf{v}_1), \quad (3, \mathbf{v}_2), \quad (-1, \mathbf{v}_3).$$

Let $P = \mathbf{aug}(\mathbf{v}_2, \mathbf{v}_1, \mathbf{v}_3)$. Display the answer for $P^{-1}AP$ [20%]. Justify your claim [5%].

☐ [25%] Ch6(c): Assume A is a given 4×4 matrix with eigenvalues $2 \pm 3i$, $1 \pm \sqrt{3}i$. Find the eigenvalues of $3A - 2I$, where I is the identity matrix.

☐ [25%] Ch6(d): Let 3×3 matrices A and B be related by $AQ = QB$ for some invertible matrix Q. Prove that the roots of the characteristic equations of A and B are identical.

☐ [25%] Ch6(e): Let A be a 2×2 matrix with eigenpairs

$$(\lambda_1, \mathbf{v}_1), \quad (\lambda_2, \mathbf{v}_2).$$

Display Fourier's model for the matrix A.

Ch6(a) $(2-\lambda)(5-\lambda)(5-\lambda)(3-\lambda)=0$ with roots $\boxed{2, 5, 5, 3}$

Ch6(b) $D = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & -1 \end{pmatrix}$ has eigenvalues $3, 5, -1$ in the correct order to give $AP = PD$, for $P = \text{aug}(v_2, v_1, v_3)$.

Ch6(c) $\det(3A - 2I - \lambda I) = \det(3I)\det(A - \frac{2}{3}I - \frac{\lambda}{3}I)$
$= 81 \det(A - (\frac{2}{3} + \frac{\lambda}{3})I)$

Then $\frac{2}{3} + \frac{\lambda}{3} =$ eigenvalue of A implies $\lambda = \boxed{\frac{4}{9} \pm i, \frac{1}{9} \pm \frac{\sqrt{3}}{3}i}$

Ch6(d) $\det(A - \lambda I) = \det(QBQ^{-1} - \lambda QQ^{-1})$
$= \det Q \det(B - \lambda I) \det Q^{-1}$
$= \det(B - \lambda I)$ because $\det Q \det Q^{-1} = \det I = 1$

Ch6(e) $\vec{x} = c_1 \vec{v}_1 + c_2 \vec{v}_2 \Rightarrow A\vec{x} = c_1 \lambda_1 \vec{v}_1 + c_2 \lambda_2 \vec{v}_2$

Staple this page to the top of all Ch6 work. Submit one package per chapter.

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Differential Equations and Linear Algebra 2250-2

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Ch7. (Linear Systems of Differential Equations)

☐ [50%] Ch7(a): Apply the eigenanalysis method to solve the system $\mathbf{x}' = A\mathbf{x}$, given

$$A = \begin{bmatrix} -4 & 1 & 1 \\ 1 & -4 & 1 \\ 0 & 0 & -4 \end{bmatrix}$$

☐ [25%] Ch7(b): Solve for $x(t)$ in the system below. Don't solve for $y(t)$!

$$\begin{aligned} x' &= x + y, \\ y' &= -9x + y. \end{aligned}$$

☐ [25%] Ch7(c): Let A be a 4×4 real matrix and assume Fourier's model is valid for A. Display the general solution $\mathbf{x}(t)$ for $\mathbf{x}' = A\mathbf{x}$ in terms of the ingredients of Fourier's model.

☐ [25%] Ch7(d): Consider a 4×4 system $\mathbf{x}' = A\mathbf{x}$. Assume A has an eigenvalue $\lambda = -1/7$ with corresponding eigenvectors

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} 1 \\ 1 \\ -1 \\ 1 \end{pmatrix}.$$

Find a nonzero solution of the differential equation with limit zero at infinity.

☐ [25%] Ch7(e): Let $x(t)$ and $y(t)$ be the amounts of salt in brine tanks A and B, respectively. Assume fresh water enters A at rate $r = 5$ gallons/minute. Let A empty to B at rate r, and let B empty at rate r. Assume the model

$$\begin{cases} x'(t) = -\dfrac{r}{50}x(t), \\ y'(t) = \dfrac{r}{50}x(t) - \dfrac{r}{100}y(t), \\ x(0) = 0, \quad y(0) = 10. \end{cases}$$

Find the maximum amount of salt ever in tank B.

(a) $\lambda_1 = -3, \ \lambda_2 = -4, \ \lambda_3 = -5$ $\quad V_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \ V_2 = \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}, \ V_3 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$

$$X(t) = c_1 e^{-3t} V_1 + c_2 e^{-4t} V_2 + c_3 e^{-5t} V_3$$

(b) $X'' = X' + y' = X' - 9X + y = X' - 9X + X' - X \Rightarrow X'' - 2X' + 10X = 0$

$r^2 - 2r + 10 = 0 \Rightarrow r_{1,2} = 1 \pm 3i \Rightarrow \boxed{X(t) = e^{t}\left(c_1 \cos(3t) + c_2 \sin(3t)\right)}$

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Ch 7 KEY continued

(c) $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ eigenvalues, v_1, v_2, v_3, v_4 eigenvectors

$$X(t) = c_1 e^{\lambda_1 t} v_1 + c_2 e^{\lambda_2 t} v_2 + c_3 e^{\lambda_3 t} v_3 + c_4 e^{\lambda_4 t} v_4$$

(d) Any nontrivial linear combination of $\begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix} e^{-1/7 t}$ and $\begin{bmatrix} 1 \\ 1 \\ -1 \\ 1 \end{bmatrix} e^{-1/7 t}$.

When $t \to \infty$, $e^{-1/7 t} \to 0 \Rightarrow$ solutions $\to 0$.

(e) By uniqueness, $X(t) = 0$. $\Rightarrow y'(t) = -\frac{r}{100} y(t)$

$$\Rightarrow y(t) = 10 e^{-\frac{r}{100} t}$$

Max $y = 10$.

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Ch10. (Laplace Transform Methods)

It is assumed that you have memorized the basic 4-item Laplace integral table and know the 6 basic rules for Laplace integrals. No other tables or theory are required to solve the problems below. If you don't know a table entry, then leave the expression unevaluated for partial credit.

☐ [35%] Ch10(a): Apply Laplace's method to the system to find a formula for $\mathcal{L}(y(t))$. Find a 2×2 system for $\mathcal{L}(x)$, $\mathcal{L}(y)$ [20%]. Solve it **only** for $\mathcal{L}(y)$ [15%]. To save time, **do not** solve for $x(t)$ or $y(t)$!

$$\begin{aligned} x'' &= 3x + 3y + 2, \\ y'' &= 4x + 2y, \\ x(0) &= 0, \quad x'(0) = 2, \\ y(0) &= 0, \quad y'(0) = 3. \end{aligned}$$

☐ [35%] Ch10(b): Solve for $x(t)$, given

$$\mathcal{L}(x(t)) = \frac{d}{ds}\left(\mathcal{L}(e^{2t}\sin 2t)\right) + \frac{s+1}{(s+2)^2} + \frac{2+s}{s^2+5s} + \mathcal{L}(t + \sin t)|_{s \to (s-2)}.$$

☐ [30%] Ch10(c): Find $f(t)$ by partial fraction methods, given

$$\mathcal{L}(f(t)) = \frac{8s^2 - 24}{(s-1)(s+3)(s+1)^2}.$$

☐ [30%] Ch10(d): Apply Laplace's method to find a formula for $\mathcal{L}(x(t))$. To save time, **do not** solve for $x(t)$! Document steps by reference to tables and rules.

$$x^{iv} - x'' = 3t^2 + 4e^{-2t} + 5e^t \sin 2t, \quad x(0) = x'(0) = x''(0) = 0, \quad x'''(0) = -1.$$

(a)
$$\begin{cases} s^2\mathcal{L}(x) - 2 = 3\mathcal{L}(x) + 3\mathcal{L}(y) + \frac{2}{s} \\ s^2\mathcal{L}(y) - 3 = 4\mathcal{L}(x) + 2\mathcal{L}(y) \end{cases} \Rightarrow \mathcal{L}(y) = \frac{\begin{vmatrix} s^2-3 & 2+\frac{2}{s} \\ -4 & 3 \end{vmatrix}}{\begin{vmatrix} s^2-3 & -3 \\ -4 & s^2-2 \end{vmatrix}}$$

(b)
$$x = x_1 + x_2 + x_3 + x_4$$
$$x_1 = -t e^{2t}\sin 2t$$
$$x_2 = e^{-2t} - t e^{-2t}$$
$$x_3 = \frac{2}{5} + \frac{3}{5}e^{-5t}$$
$$x_4 = e^{2t}(t + \sin t)$$

$$\mathcal{L}(-t e^{2t}\sin 2t)$$
$$\frac{s+1}{(s+2)^2} = \frac{1}{s+2} - \frac{1}{(s+2)^2}$$
$$\frac{2+s}{s^2+5s} = \frac{2}{5}\cdot\frac{1}{s} + \frac{3}{5}\cdot\frac{1}{s+5}$$

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Ch10 KEY continued

(c) $$\frac{8s^2-24}{(s-1)(s+3)(s+1)^2} = \frac{4}{s+1} + \frac{4}{(s+1)^2} - \frac{1}{s-1} - \frac{3}{s+3}$$

$$f(t) = 4e^{-t} + 4te^{-t} - e^{t} - 3e^{-3t}$$

(d) $$s^4\mathcal{L}(x) - (-1) - s^2\mathcal{L}(x) = \frac{6}{s^3} + \frac{4}{s+2} + \frac{10}{(s-1)^2+4}$$

$$\mathcal{L}(x) = \frac{1}{s^4-s^2}\left[\frac{6}{s^3} + \frac{4}{s+2} + \frac{10}{(s-1)^2+4} - 1\right]$$