Ch3. (Linear Systems and Matrices)

☐ [50%] Ch3(a): Find the third entry on the fifth row of the inverse matrix \( B^{-1} \) by the formula 
\[
B^{-1} = \text{adj}(B)/\text{det}(B).
\]
Evaluate determinants by any method: triangular, swap, combo, multiply, cofactor. The use of \( 3 \times 3 \) Sarrus’ rule is disallowed (\( 2 \times 2 \) use is allowed).

\[
B = \begin{bmatrix}
1 & 1 & -2 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 \\
0 & -1 & 2 & 0 & 0 \\
1 & 0 & 0 & 3 & 0 \\
-1 & 0 & 1 & 3 & 4
\end{bmatrix}
\]

☐ [50%] Ch3(b): Determine all values of \( k \) such that the system \( Rx = f \) has (1) infinitely many solutions [15%], (2) a unique solution [15%] and (3) no solution [20%].

\[
R = \begin{bmatrix}
2 & 1 & k \\
2 & k & -2 \\
0 & 0 & 4
\end{bmatrix}, \quad f = \begin{pmatrix}
0 \\
1 + k
\end{pmatrix}
\]

☐ [25%] Ch3(c): Let \( A \) be a \( 3 \times 3 \) triangular matrix with diagonal entries 3, 1, 0. Prove that \( Ax = 0 \) has infinitely many solutions \( x \).

☐ [25%] Ch3(d): Let \( A \) denote a \( 4 \times 3 \) nonzero matrix. Find an example \( A \) such that \( Ax = 0 \) has a unique solution.

☐ [25%] Ch3(e): There are real \( 2 \times 2 \) matrices \( A \) such that \( A^2 = -I \), where \( I \) is the identity matrix. Display one such matrix \( A \) and justify the claim.

\[
\text{ch3@ \quad } \text{Cof}(B, 3, 5) = (-1)^8 \begin{vmatrix}
1 & 2 & 0 \\
1 & 0 & 2 \\
0 & 1 & 3
\end{vmatrix} = 12, \quad \text{det}(B) = 24 \quad \frac{1}{2}
\]

\[
\text{ch2@ \quad } \text{Aug}(R, f) = \begin{pmatrix}
2 & 0 & 1 & f \\\n0 & -1 & 0 & 1 + k
\end{pmatrix}
\]

\( 1 \) never \( \infty - \text{many} \) sol, \( 2 \) \( k \neq 0 \) unique sol, \( 3 \) \( k = 1 \) single eq

\[
\text{ch3@ Because } \text{det}(A) = 0, \text{then } \infty - \text{many sols}
\]

\[
\text{ch3@ } A = \begin{pmatrix}
1 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}
\]

unique sol \( \overline{x} = \overline{0} \) + \( A \overline{x} = \overline{0} \).

\[
\text{ch3@ Let } \quad A = \begin{pmatrix}
0 & 1 \\
-1 & 0
\end{pmatrix}, \text{ then } A^2 = -I
\]

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Ch4. (Vector Spaces)

☐ [40%] Ch4(a): State an RREF test (not a determinant test) to detect the independence or dependence of fixed vectors $v_1, v_2, v_3$ in $\mathbb{R}^5$ [10%]. Apply the test to the vectors below [25%]. Report independent or dependent [5%].

$$v_1 = \begin{pmatrix} -1 \\ 1 \\ 2 \\ 0 \\ 1 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 3 \\ 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}, \quad v_3 = \begin{pmatrix} 4 \\ -1 \\ -1 \\ 0 \\ 1 \end{pmatrix}.$$  

☐ [60%] Ch4(b): Define $V$ to be the set of all vectors $x$ in $\mathbb{R}^5$ such that $2x_1 + x_5 = x_3$ and $c \cdot x = 0$, where $c$ has constant components $c_1$ through $c_5$. Prove that $V$ is a subspace of $\mathbb{R}^5$.

☐ [60%] Ch4(c): Find a basis of fixed vectors in $\mathbb{R}^4$ for (1) the column space of the $4 \times 4$ matrix $A$ below [30%] and (2) the row space of the $4 \times 4$ matrix $A$ below [30%]. The reported basis must consist of columns of $A$ and rows of $A$, respectively.

$$A = \begin{pmatrix} 1 & 1 & -1 & 0 \\ 1 & 1 & -1 & -4 \\ 0 & 0 & 0 & 5 \\ 2 & 2 & -2 & 6 \end{pmatrix}.$$  

☐ [40%] Ch4(d): Find a $4 \times 4$ system of linear equations for the constants $a$, $b$, $c$, $d$ in the partial fractions decomposition below [10%]. Solve for $a$, $b$, $c$, $d$, showing all RREF steps [25%]. Report the answers [5%].

$$\frac{3x^2 - 14x + 3}{(x + 1)^2(x - 2)^2} = \frac{a}{x+1} + \frac{b}{(x+1)^2} + \frac{c}{x-2} + \frac{d}{(x-2)^2}.$$  

Ch4@ Test: $v_1, v_2, v_3$ independent in $\mathbb{R}^5$  $\Rightarrow$ rank($\text{aug}(v_1, v_2, v_3)$) = 3.
Apply: $\text{aug}(v_1, v_2, v_3)$ has 2 leading ones in RREF $\Rightarrow$ rank = 3 $\Rightarrow$ indip.

Ch4@ Let $A$ have rows $(2, 0, -1, 0, 1)$, $\frac{1}{2}$, $\frac{1}{2}$, $\frac{1}{2}$, $\frac{1}{2}$. Then the restriction eqs can be written as $A \vec{x} = \vec{0}$. Apply Theorem 2 E&F. Then $V$ is a subspace.

Ch4@ Find $\text{rref}(A)$, $\text{rref}(A^T)$. By the Pivot Theorem, $\begin{pmatrix} 1, 4 \end{pmatrix}$ = independent cols of $A$ $\begin{pmatrix} 1, 2 \end{pmatrix}$ = independent rows of $A$.

Ch4@ $a = -20/27$, $b = 20/9$, $c = 20/27$, $d = -13/9$.

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Ch5. (Linear Equations of Higher Order)

[25%] Ch5(a): Using the recipe for higher order constant-coefficient differential equations, write out the general solutions of the differential equations whose characteristic equations are given below.

1. \[ r^2(r^2 - 5r + 25) = 0, \]
2. \[ (r - 4)^2(r^2 + 2r + 3)(r^2 - 16)^3 = 0 \]

[25%] Ch5(b): Given a damped spring-mass system \( mx''(t) + cx'(t) + kx(t) = 0 \) with \( m = 10, c = 13 \) and \( k = 4 \), solve the differential equation [25%] and classify the answer as over-damped, critically damped or under-damped [5%].

[50%] Ch5(c): Determine the correct trial solution for \( y_p \) according to the method of undetermined coefficients. Do not evaluate the undetermined coefficients!

\[ y'' + 9y'' = x(x + 2e^{3x}) + 3x \cos 3x + 4e^{-3x} \]

[25%] Ch5(d): Find the steady-state periodic solution for the equation

\[ x'' + 4x' + 29x = \cos(3t) \]

Ch5(a)
1. Roots = \( 0, 0, 0, 5, 5, 5 \), \( \gamma = u_1 e^{0x} + u_2 e^{5x} + u_3 e^{-5x} \),
   \[ u_1 = c_1 + c_2 x + c_3 x^2 + c_4 x^3 + c_5 x^4 \], \( u_2 = c_6 + c_7 x + c_8 x^2 \), \( u_3 = c_9 \)

2. Roots = \( 4, 4, 4, 4, 4, 4, 4 \) (12 roots), \( y = u_1 e^{4x} + u_2 e^{-4x} \cos(2x) \)
   \[ u_3 e^{4x} \sin(2x) + u_4 e^{-4x} \]
   \[ u_1 = c_1 + c_2 x + c_3 x^3 + c_4 x^2 + c_5 x^4 \], \( u_2 = c_6 + c_7 x + c_8 + c_9 x \)
   \[ u_4 = c_{10} + c_{11} x + c_{12} \]

Ch5(b) Overdamped, Roots = \( -4/5, -1/2 \), \( y(t) = c_1 e^{-4t/5} + c_2 e^{-t/2} \)

Ch5(c) \( y = y_1 + y_2 + y_3 + y_4 \), \( y_1 = \left( d_1 + d_2 x + d_3 x^2 \right) e^{3x} \), \( y_2 = \left( d_4 + d_5 x \right) e^{3x} \), \( y_3 = \left[ (d_6 + d_7 x) \cos 3x + (d_8 + d_9 x) \sin 3x \right] x \), \( y_4 = d_{10} e^{-3x} \)

Find up are equal to \( y_1, y_2, y_3 \) only.

Ch5(d) Trial solution \( x = d_1 \cos 3t + d_2 \sin 3t \) is the unique periodic solution.

\[ d_1 = \frac{20}{A}, d_2 = \frac{12}{A}, A = 1000 + 44 = 544 \] found from the equations

\[ \begin{cases} 20 d_1 + 12 d_2 = 1 \quad \text{(simplified)} \quad d_1 = \frac{5}{136} \\ -12 d_1 + 30 d_2 = 0 \end{cases} \]

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Ch6. (Eigenvalues and Eigenvectors)

☐ [20%] Ch6(a): Find the eigenvalues of the matrix $A$:

$$A = \begin{bmatrix} 2 & 4 & -1 & 0 \\ 0 & 5 & -2 & 1 \\ 0 & 0 & 4 & 1 \\ 0 & 0 & 1 & 4 \end{bmatrix}$$

☐ [25%] Ch6(b): Let $A$ be a $3 \times 3$ matrix with eigenpairs $(5, v_1)$, $(3, v_2)$, $(-1, v_3)$. Let $P = \text{aug}(v_2, v_1, v_3)$. Display the answer for $P^{-1}AP$ [20%]. Justify your claim [5%].

☐ [25%] Ch6(c): Assume $A$ is a given $4 \times 4$ matrix with eigenvalues $2 \pm 3i$, $1 \pm \sqrt{3}i$. Find the eigenvalues of $3A - 2I$, where $I$ is the identity matrix.

☐ [25%] Ch6(d): Let $3 \times 3$ matrices $A$ and $B$ be related by $AQ = QB$ for some invertible matrix $Q$. Prove that the roots of the characteristic equations of $A$ and $B$ are identical.

☐ [25%] Ch6(e): Let $A$ be a $2 \times 2$ matrix with eigenpairs $(\lambda_1, v_1)$, $(\lambda_2, v_2)$.

Display Fourier's model for the matrix $A$.

Ch6(A) $$(2 - \lambda)(5 - \lambda)(3 - \lambda) = 0 \quad \text{with roots} \quad 2, 5, 3$$

Ch6(B) $D = \begin{bmatrix} 2 & 3 & 0 \\ 3 & 0 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ has eigenvalues $3, 5, -1$. In the correct order to give $AP = PDP^{-1}$, for $P = \text{aug}(v_2, v_1, v_3)$.

Ch6(C) $\det(3A - 2I - \lambda I) = \det(3I) \det(A - \frac{2}{3}I - \frac{\lambda}{3}I)$

Then $\frac{2}{3} + \frac{1}{3} = \text{eigenvalue} \lambda$ of $A$ implies $\lambda = \frac{4}{3} + \frac{1}{3} = \frac{5}{3}$.

Ch6(D) $\det(A - \lambda I) = \det(QAQ^{-1} - \lambda QQ^{-1})$

$= \det(Q) \det(B - \lambda I) \det Q^{-1}$

$= \det(B - \lambda I)$ because $\det(Q) \det(Q^{-1}) = \det I = 1$

Ch6(E) $\hat{x} = c_1 v_1 + c_2 v_2 \implies A\hat{x} = c_1 \lambda_1 v_1 + c_2 \lambda_2 v_2$

Staple this page to the top of all Ch6 work. Submit one package per chapter.
Ch7. (Linear Systems of Differential Equations)

☐ [50%] Ch7(a): Apply the eigenanalysis method to solve the system $x' = Ax$, given

$$A = \begin{bmatrix} -4 & 1 & 1 \\ 1 & -4 & 1 \\ 0 & 0 & -4 \end{bmatrix}$$

☐ [25%] Ch7(b): Solve for $x(t)$ in the system below. Don’t solve for $y(t)$!

$$x' = x + y,$$
$$y' = -9x + y.$$ 

☐ [25%] Ch7(c): Let $A$ be a $4 \times 4$ real matrix and assume Fourier’s model is valid for $A$. Display the general solution $x(t)$ for $x' = Ax$ in terms of the ingredients of Fourier’s model.

☐ [25%] Ch7(d): Consider a $4 \times 4$ system $x' = Ax$. Assume $A$ has an eigenvalue $\lambda = -1/7$ with corresponding eigenvectors

$$v_1 = \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 1 \\ 1 \\ -1 \\ 1 \end{pmatrix}.$$ 

Find a nonzero solution of the differential equation with limit zero at infinity.

☐ [25%] Ch7(e): Let $x(t)$ and $y(t)$ be the amounts of salt in brine tanks $A$ and $B$, respectively. Assume fresh water enters $A$ at rate $r = 5$ gallons/minute. Let $A$ empty to $B$ at rate $r$, and let $B$ empty at rate $r$. Assume the model

$$\begin{align*}
x'(t) &= -\frac{r}{50} x(t), \\
y'(t) &= \frac{r}{50} x(t) - \frac{r}{100} y(t), \\
x(0) &= 0, \quad y(0) = 10.
\end{align*}$$

Find the maximum amount of salt ever in tank $B$.

(a) $\lambda_1 = -3, \quad \lambda_2 = -4, \quad \lambda_3 = -5, \quad v_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad v_2 = \begin{pmatrix} -1 \\ 1 \\ -1 \\ 1 \end{pmatrix}, \quad v_3 = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}$

$$x(t) = c_1 e^{-3t} v_1 + c_2 e^{-4t} v_2 + c_3 e^{-5t} v_3$$

(b) $x'' = x' + y' = x' - 9x + y = x' - 9x + x' = x' - 2x + 10x = 0$

$$r^2 - 2r + 10 = 0 \Rightarrow r_{1,2} = 1 \pm 3i \Rightarrow x(t) = e^{t} \left(c_1 \cos(3t) + c_2 \sin(3t)\right)$$

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Chapter 7: Key

(c) \( \lambda_1, \lambda_2, \lambda_3, \lambda_4 \) eigenvalues, \( v_1, v_2, v_3, v_4 \) eigenvectors

\[ X(t) = c_1 e^{\lambda_1 t} v_1 + c_2 e^{\lambda_2 t} v_2 + c_3 e^{\lambda_3 t} v_3 + c_4 e^{\lambda_4 t} v_4 \]

(d) Any nontrivial linear combination of \( \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix} e^{\frac{1}{2} t} \) and \( \begin{bmatrix} 1 \\ 1 \\ -1 \\ 1 \end{bmatrix} e^{\frac{1}{2} t} \).

When \( t \to \infty \), \( e^{\frac{1}{2} t} \to 0 \Rightarrow \) solutions \( \to 0 \).

(e) By uniqueness, \( X(t) = 0 \). \( \Rightarrow \) \( y'(t) = -\frac{r}{100} y(t) \)

\( \Rightarrow \) \( y(t) = 10 e^{\frac{-r}{100} t} \)

Max \( y = 10 \).
Ch10. (Laplace Transform Methods)

It is assumed that you have memorized the basic 4-item Laplace integral table and know the 6 basic rules for Laplace integrals. No other tables or theory are required to solve the problems below. If you don’t know a table entry, then leave the expression unevaluated for partial credit.

[35%] Ch10(a): Apply Laplace’s method to the system to find a formula for \( \mathcal{L}(y(t)) \). Find a 2 \( \times \) 2 system for \( \mathcal{L}(x), \mathcal{L}(y) \) [20%]. Solve it only for \( \mathcal{L}(y) \) [15%]. To save time, do not solve for \( x(t) \) or \( y(t) \)!

\[
\begin{align*}
x'' &= 3x + 3y + 2, \\
y'' &= 4x + 2y, \\
x(0) &= 0, \quad x'(0) = 2, \\
y(0) &= 0, \quad y'(0) = 3.
\end{align*}
\]

[35%] Ch10(b): Solve for \( x(t) \), given

\[
\mathcal{L}(x(t)) = \frac{d}{ds} \left( \mathcal{L}(e^{2t} \sin 2t) \right) + \frac{s + 1}{(s + 2)^2} + \frac{2 + s}{s^2 + 5s} + \mathcal{L}(t + \sin t) |_{s \to (s - 2)}.
\]

[30%] Ch10(c): Find \( f(t) \) by partial fraction methods, given

\[
\mathcal{L}(f(t)) = \frac{8s^2 - 24}{(s - 1)(s + 3)(s + 1)^2}.
\]

[30%] Ch10(d): Apply Laplace’s method to find a formula for \( \mathcal{L}(x(t)) \). To save time, do not solve for \( x(t) \)!

Document steps by reference to tables and rules.

\[
x''' - x'' = 3t^2 + 4e^{-2t} + 5e^t \sin 2t, \quad x(0) = x'(0) = x''(0) = 0, \quad x'''(0) = -1.
\]

(a) \[
\begin{align*}
8^2 \mathcal{L}(x) - 2 &= 3 \mathcal{L}(x) + 3 \mathcal{L}(y) + \frac{2}{s} \\
8^2 \mathcal{L}(y) - 3 &= 4 \mathcal{L}(x) + 2 \mathcal{L}(y)
\end{align*}
\]

\[
\Rightarrow \mathcal{L}(y) = \begin{vmatrix}
8^2 - 3 & 2 + \frac{2}{s} \\
-4 & 3
\end{vmatrix}
\]

(b) \[
x = x_1 + x_2 + x_3 + x_4
\]

\[
\begin{align*}
x_1 &= -t \ e^{2t} \sin 2t \\
x_2 &= e^{2t} - t \ e^{2t} \\
x_3 &= \frac{2}{5} + \frac{3}{5} \ e^{-5t} \\
x_4 &= e^{2t} \ (t + \sin t)
\end{align*}
\]

\[
\mathcal{L}(-t \ e^{2t} \sin 2t)
\]

\[
\begin{align*}
\frac{s + 1}{(s + 2)^2} &= \frac{1}{s + 2} - \frac{1}{(s + 2)^2} \\
\frac{2 + 5}{s^2 + 5s} &= \frac{2}{5} \cdot \frac{1}{s} + \frac{3}{5} \cdot \frac{1}{s + 5}
\end{align*}
\]

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(C) \[ \frac{8s^2 - 24}{(s-1)^6(s+3)(s+1)^2} = \frac{4}{s+1} + \frac{4}{(s+1)^2} - \frac{1}{s-1} - \frac{3}{s+3} \]

\[ f(t) = 4e^{-t} + 4te^{-t} - e^t - 3e^{-3t} \]

(d) \[ s^4 \mathcal{L}(x) - (-1)^2 \mathcal{L}(x) = \frac{6}{s^3} + \frac{4}{s+2} + \frac{10}{(s-1)^2+4} \]

\[ \mathcal{L}(x) = \frac{1}{s^4-s^2} \left[ \frac{6}{s^3} + \frac{4}{s+2} + \frac{10}{(s-1)^2+4} - 1 \right] \]