Chapter 1

Fundamentals

Contents

Introduced here are notation, definitions and background results suitable for use in differential equations.

Prerequisites include college algebra, coordinate geometry, differential calculus and integral calculus. The examples and exercises include a review of some calculus topics, especially derivatives, integrals, numerical integration, hand and computer graphing. A significant part of the review is algebraic manipulation of logarithms, exponentials, sines and cosines.

New topics of an elementary nature are introduced. The chapter starts immediately with applications to differential equations that require only a background from pre-calculus in exponential and logarithmic functions. No differential equations background is assumed or used.

Differential equations are defined and insight is given into the notion of *answer* for differential equations in science and engineering.

Basic topics included here are direction fields, phase line diagrams and bifurcation diagrams, which require only a calculus background. Applications of these ideas appear later in the text, after more solution methods have been introduced.

Advanced topics include existence-uniqueness theory and implicit functions. Included are some practical methods for employing computer algebra systems to assist with finding solutions, verifying equations, modeling, and related topics.

1.1 Exponential Modeling

The model differential equation $y' = ky$ is studied through a variety of specific applications. All applications use the exponential solution $y = y_0 e^{kt}$.

Three Examples

These applications are studied:

Growth–Decay Models
Newton Cooling
Verhulst Logistic Model

It is possible to solve a variety of differential equations without reading this book or any other differential equations text. Given in the table below are three exponential models and their known solutions, all of which will be derived from principles of elementary differential calculus.

Growth-Decay	$\frac{dA}{dt} = kA(t),\ A(0) = A_0$
	$\boxed{A(t) = A_0 e^{kt}}$
Newton Cooling	$\frac{du}{dt} = -h(u(t) - u_1),\ u(0) = u_0$
	$\boxed{u(t) = u_1 + (u_0 - u_1)e^{-ht}}$
Verhulst Logistic	$\frac{dP}{dt} = (a - bP(t))P(t),\ P(0) = P_0$
	$\boxed{P(t) = \frac{aP_0}{bP_0 + (a - bP_0)e^{-at}}}$

These models and their solution formulas form a foundation of intuition for all of differential equation theory. Considerable use will be made of the models and their solution formulas.

The *physical meanings* of the constants k, A_0, h, u_1, u_0, a, b, P_o and the variable names $A(t)$, $u(t)$, $P(t)$ are given below, as each example is discussed.

Background

Mathematical background used in exponential modeling is limited to algebra and basic calculus. The following facts are assembled for use in applications.

$\ln e^x = x,\ e^{\ln y} = y$	In words, the exponential and the logarithm are inverses. The domains are $-\infty < x < \infty$, $0 < y < \infty$.
$e^0 = 1,\ \ln(1) = 0$	Special values, usually memorized.
$e^{a+b} = e^a e^b$	In words, the exponential of a sum of terms is the product of the exponentials of the terms.
$(e^a)^b = e^{ab}$	Negatives are allowed, e.g., $(e^a)^{-1} = e^{-a}$.
$\left(e^{u(t)}\right)' = u'(t)e^{u(t)}$	The *chain rule* of calculus implies this formula from the identity $(e^x)' = e^x$.
$\ln AB = \ln A + \ln B$	In words, the logarithm of a product of factors is the sum of the logarithms of the factors.
$B\ln(A) = \ln\left(A^B\right)$	Negatives are allowed, e.g., $-\ln A = \ln \dfrac{1}{A}$.
$(\ln\lvert u(t)\rvert)' = \dfrac{u'(t)}{u(t)}$	The identity $(\ln(x))' = 1/x$ implies this general version by the *chain rule*.

Applied topics using exponentials inevitably lead to equations involving logarithms. Conversion of exponential equations to logarithmic equations, and the reverse, happens to be an important subtopic of differential equations. The examples and exercises contain typical calculations.

Growth-Decay Equation

Growth and decay models in science are based upon the exponential equation

$$y = y_0 e^{kx}. \tag{1}$$

The exponential e^{kx} increases if $k > 0$ and decreases if $k < 0$. Models based upon exponentials are called **growth models** if $k > 0$ and **decay models** if $k < 0$. Examples of growth models include population growth and compound interest. Examples of decay models include radioactive decay, radiocarbon dating and drug elimination. Typical growth and decay curves appear in Figure 1.

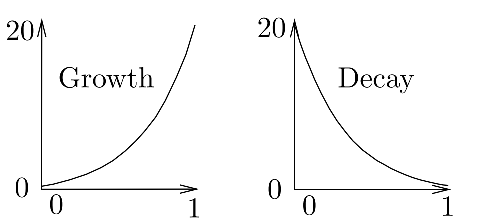


Figure 1. Growth and decay curves.

Definition 1 (Growth-Decay Equation)
The differential equation

$$\frac{dy}{dx} = ky \tag{2}$$

is called a **growth-decay** differential equation.

A solution of (2) is given by (1); see the verification on page 9. It is possible to show directly that the differential equation has no other solutions, hence the terminology *the solution* $y = y_0 e^{kx}$ is appropriate; see the verification on page 10. The solution $y = y_0 e^{kx}$ in (1) satisfies the **growth-decay initial value problem** :

$$\frac{dy}{dx} = ky, \quad y(0) = y_0. \tag{3}$$

Recipe for Solving a Growth-Decay Equation. Numerous applications to first order differential equations are based upon equations that have the general form $y' = ky$. Whenever this form is encountered, immediately the solution is known: $y = y_0 e^{kx}$.

The report of the answer without *solving* the differential equation is called a **recipe** for the solution. The recipe for $y' = ky$ has an immediate generalization to the second order differential equations which are studied in electrical circuits and mechanical systems.

Newton Cooling Equation

If a fluid is held at constant temperature, then the cooling of a body immersed in the fluid is subject to **Newton's cooling law** :

> The rate of temperature change of the body is proportional to the difference between the body's temperature and the fluid's constant temperature.

Translation to mathematical notation gives the differential equation

$$\frac{du}{dt} = -h(u(t) - u_1) \tag{4}$$

where $u(t)$ is the temperature of the body, u_1 is the constant ambient temperature of the fluid and $h > 0$ is a constant of proportionality.

A typical instance is the cooling of hot chocolate in a room. Here, u_1 is the wall thermometer reading and $u(t)$ is the reading of a dial thermometer immersed in the chocolate drink.

Theorem 1 (Solution of Newton's Cooling Equation)
The change of variable $y(t) = u(t) - u_1$ translates the cooling equation $du/dt = -h(u - u_1)$ into the growth-decay equation $y'(t) = -hy(t)$. Therefore, the cooling solution is given in terms of $u_0 = u(0)$ by the equation

$$u(t) = u_1 + (u_0 - u_1)e^{-ht}. \tag{5}$$

The result is proved on page 10. It shows that a cooling model is just a translated growth-decay model. The solution formula (5) can be expressed in words as follows:

> The dial thermometer reading of the hot chocolate equals the wall thermometer reading plus an exponential decay term.

Cooling problems have *curious extra conditions*, usually involving physical measurements, for example the three equations

$$u(0) = 100, \quad u(1) = 90 \quad \text{and} \quad u(\infty) = 22.$$

The extra conditions implicitly determine the actual values of the three undetermined parameters h, u_1, u_0. The logic is as follows. Equation (5) is a relation among 5 variables. Substitution of values for t and u eliminates 2 of the 5 variables and gives an equation for u_1, u_0, h. The system of three equations in three unknowns can be solved for the actual values of u_1, u_0, h.

Stirring Effects. Exactly how to maintain a constant ambient temperature is not addressed by the model. One method is to stir the liquid, as in Figure 2, but the mechanical energy of the stirrer will inevitably appear as heat in the liquid. In the simplest case, stirring effects add a fixed constant temperature S_0 to the ambient temperature u_1. For slow stirring, $S_0 = 0$ is assumed, which is the above model.

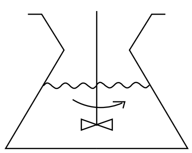


Figure 2. Flask Cooling with Stirring.

Populations

The **human population** of the world reached six billion in 1999, according to the U.S. Census Bureau.

World Population Estimate
12/1999
6,033,366,287
Source: U.S. Census Bureau

The term **population** refers to humans. In literature, it may also refer to bacteria, insects, rodents, rabbits, wolves, trees, yeast and similar living things that have birth rates and death rates.

Malthusian Population Equation. A constant birth rate or a constant death rate is unusual in a population, but these ideal cases have been studied. The biological reproduction law is called **Malthus's law** :

> The population flux is proportional to the population itself.

This biological law can be written in calculus terms as

$$\frac{dP}{dt} = kP(t)$$

where $P(t)$ is the population count at time t. The reasoning is that **population flux** is the expected change in population size for a unit change in t, or in the limit, dP/dt. A careful derivation of such calculus laws from English appears in Example 6 on page 683.

The theory of growth-decay differential equations implies that population studies based upon Malthus's law employ the exponential model

$$P(t) = P_0 e^{k(t-t_0)}.$$

The number k is the difference of the birth and death rates, or **combined birth-death rate** , t_0 is the **initial time** and P_0 is the **initial population size** at time $t = t_0$.

Verhulst Logistic Equation. The population model $P' = kP$ was studied around 1840 by the Belgian demographer and mathematician Pierre-Francois Verhulst (1804–1849) in the special case when k depends on the population size $P(t)$. Under Verhulst's assumptions, $k = a - bP$ for positive constants a and b, so that $k > 0$ (growth) for populations smaller than a/b and $k < 0$ (decay) when the population exceeds a/b. The result is called the **logistic equation** :

$$P' = (a - bP)P. \tag{6}$$

Verhulst established the limit formula

$$\lim_{t\to\infty} P(t) = a/b, \tag{7}$$

which has the interpretation that initial populations $P(0)$, regardless of size, will after a long time stabilize to size approximately a/b. The constant a/b is called the **carrying capacity** of the population.

Limit formula (7) follows directly from solution formula (8) below.

Theorem 2 (Verhulst Logistic Solution)
The change of variable $y(t) = P(t)/(a - bP(t))$ transforms the logistic equation $P'(t) = (a - bP(t))P(t)$ into the growth-decay equation $y'(t) = ay(t)$. Then the logistic equation solution is given by

$$P(t) = \frac{aP(0)}{bP(0) + (a - bP(0))e^{-at}}. \tag{8}$$

The derivation appears on page 11. The viewpoint of the result is that a logistic model is obtained from a growth-decay model by a fractional change of variable. When $b = 0$, the logistic model and the growth-decay model are the same and formula (8) reduces to the solution of growth-decay equation $y' = ay$. The recipe formula for the solution remains valid regardless of the signs of a and b, provided the quotient is defined.

Examples

1 **Example (Growth-Decay Recipe)** Solve the initial value problem $y' = 2y$, $y(0) = 4$.

Solution: This is a growth-decay equation $y' = ky$, $y(0) = y_0$ with $k = 2$, $y_0 = 4$. Therefore, the solution is $y = 4e^{2x}$. No method is required to *solve* the equation $y' = 2y$, because of the theory on page 3.

2 **Example (Newton Cooling Recipe)** Solve the initial value problem $u' = -3(u - 72)$, $u(0) = 190$.

Solution: This is a Newton cooling equation $u' = -h(u - u_1)$, $u(0) = u_0$ with $h = 3$, $u_1 = 72$, $u_0 = 190$. Therefore, the solution is $u(t) = 72 + 118e^{-3t}$. No method is required to *solve* the equation $u' = -3(u - 72)$, because of the theorem on page 4.

3 **Example (Verhulst Logistic Recipe)** Solve the initial value problem $P' = (1 - 2P)P$, $P(0) = 500$.

Solution: This is a Verhulst logistic equation $P' = (a - bP)P$, $P(0) = P_0$ with $a = 1$, $b = 2$, $P_0 = 500$. Therefore, the solution is $P(t) = \dfrac{500}{1000 - 999e^{-t}}$. No method is required to *solve* the equation $P' = (1 - 2P)P$, because of the theorem on page 6.

4 **Example (Standing Room Only)** Justify the estimate 2600 for the year in which each human has only one square foot of land to stand upon. Assume the Malthus model $P(t) = 3.34e^{0.02(t-1965)}$, with t in years and P in billions.

Solution: The mean radius of the earth is 3965 miles or $20,935,200$ feet. The surface area formula $4\pi r^2$ gives $5,507,622$ billion square feet. About 20% of this is land, or $1,101,524$ billion square feet.

The estimate 2600 is obtained by solving for t years in the equation

$$3.34e^{0.02(t-1965)} = 1101524.$$

The college algebra details:

$e^{0.02(t-1965)} = \dfrac{1101524}{3.34}$	Isolate the exponential on the left. Solving for t.

$\ln e^{0.02(t-1965)} = \ln 329797.6$	Simplify the right side and take the logarithm of both sides.
$0.02(t-1965) = 12.706234$	On the right, compute the logarithm. Use $\ln e^u = u$ on the left.
$t = 1965 + \dfrac{12.706234}{0.02}$	Solve for t.
$= 2600.3.$	About the year 2600.

5 Example (Rodent Growth) A population of two rodents in January reproduces to population sizes 20 and 110 in June and October, respectively. Determine a Malthusian law for the population and test it against the data.

Solution: However artificial this example might seem, it is almost a real experiment; see Braun [**?**], Chapter 1, and the reference to rodent *Microtus Arvallis Pall.*

The law proposed is $P = 2e^{2t/5}$, which is 40% growth, $k = 2/5$. For a 40% rate, $P(6) \approx 2e^{12/5} = 22.046353$ and $P(10) \approx 2e^{2(10)/5} = 109.1963$. The agreement with the data is reasonable. It remains to explain how this "40% law" was invented.

The Malthusian model $P(t) = P_0e^{kt}$, with t in months, fits the three data items $P(0) = 2$, $P(6) = 20$ and $P(10) = 110$ provided $P_0 = 2$, $2e^{6k} = 20$ and $2e^{10k} = 110$. The exponential equations are solved for $k = \ln(10)/6$ and $k = \ln(55)/10$, resulting in the two growth constants $k = 0.38376418$ and $k = 0.40073332$. The average growth rate is 39.2%, or about 40%.

6 Example (Flask Cooling) A flask of water is heated to 95C and then allowed to cool in ambient room temperature 21C. The water cools to 80C in three minutes. Verify the estimate of 48 minutes to reach 23C.

Solution: Basic modeling by Newton's law of cooling gives the temperature as $u(t) = u_1 + (u_0 - u_1)e^{-kt}$ where u_1, u_0 and k are parameters. Three conditions are given in the English statement of the problem.

$u(\infty) = 21$	The ambient air temperature is 21C.
$u(0) = 95$	The flask is heated at $t = 0$ to 95C.
$u(3) = 80$	The flask cools to 80C in three minutes.

In the details below, it will be shown that the parameter values are $u_1 = 21$, $u_0 - u_1 = 74$, $k = 0.075509216$.

To find u_1:

$21 = u(\infty)$	Given ambient temperature condition.
$= \lim_{t\to\infty} u(t)$	Definition of $u(\infty)$.
$= \lim_{t\to\infty} u_1 + (u_0 - u_1)e^{-kt}$	Definition of $u(t)$.
$= u_1$	The exponential has limit zero.

To calculate $A_0 = 74$ from $u(0) = 95$:

$95 = u(0)$	Given initial temperature condition.
$= u_1 + (u_0 - u_1)e^{-k(0)}$	Definition of $u(t)$ at $t = 0$.
$= 21 + A_0$	Use $e^0 = 1$.

Therefore, $A_0 = 95 - 21 = 74$.

Computation of k starts with the equation $u(3) = 80$, which reduces to $21 + 74e^{-3k} = 80$. This exponential equation is solved for k as follows:

$e^{-3k} = \dfrac{80-21}{74}$	Isolate the exponential factor on the left side of the equation.
$\ln e^{-3k} = \ln \dfrac{80-21}{74}$	Take the logarithm of both sides.
$-3k = \ln(59/74)$	Simplify the fraction. Apply $\ln e^u = u$ on the left.
$k = \dfrac{1}{3}\ln(74/59)$	Divide by -3, then on the right use $-\ln x = \ln(1/x)$.

The estimate $u(48) \approx 23$ will be verified. The time t at which $u(t) = 23$ is found by solving the equation $21 + 74e^{-kt} = 23$ for t. A checkpoint is $-kt = \ln(2/74)$, from which t is isolated on the left. After substitution of $k = 0.075509216$, the value is $t = 47.82089$.

7 Example (Baking a Roast) A beef roast at room temperature 70F is put into a 350F oven. A meat thermometer reads 100F after four minutes. Verify that the roast is done (340F) in 120 minutes.

Solution: The roast is done when the thermometer reads 340F or higher. If $u(t)$ is the meat thermometer reading after t minutes, then it must be verified that $u(120) \geq 340$.

Even though the roast is heating instead of cooling, the beef roast temperature $u(t)$ after t minutes is given by the Newton cooling equation $u(t) = u_1 + (u_0 - u_1)e^{-kt}$, where u_1, u_0 and k are parameters. Three conditions appear in the statement of the problem:

$u(\infty) = 350$	The ambient oven temperature is 350F.
$u(0) = 70$	The beef is 70F at $t = 0$.
$u(4) = 100$	The roast heats to 100F in four minutes.

As in the *flask cooling example*, page 8, the first two relations above lead to $u_1 = 350$ and $u_0 - u_1 = -280$. The last relation determines k from the equation $350 - 280e^{-4k} = 100$. Solving by the methods of the flask cooling example gives $k = \frac{1}{4}\ln(280/250) \approx 0.028332171$. Then $u(120) = 350 - 280e^{-120k} \approx 340.65418$.

Details and Proofs

Growth-Decay Equation Existence Proof. It will be verified that $y = y_0 e^{kx}$ is a solution of $y' = ky$. It suffices to expand the left side (LHS) and right side (RHS) of the differential equation and compare them for equality.

$\text{LHS} = \dfrac{dy}{dx}$	The left side of $\dfrac{dy}{dx} = ky$ is dy/dx.
$= \dfrac{d}{dx}\left(y_0 e^{kx}\right)$	Substitute $y = y_0 e^{kx}$.
$= y_0 k e^{kx}$	Apply the rule $(e^u)' = u'e^u$.
$\text{RHS} = ky$	The right side of $\dfrac{dy}{dx} = ky$ is ky.
$= k(y_0 e^{kx})$	Substitute $y = y_0 e^{kx}$.

Therefore, LHS = RHS. This completes the proof.

Growth-Decay Equation Uniqueness Proof. It will be shown that $y = y_0 e^{kx}$ is the only solution of $y' = ky$, $y(0) = y_0$. The idea is to reduce the question to the application of a result from calculus. This is done by a clever change of variables, which has been traced back to Kümmer.[1]

Assume that y is a given solution of $y' = ky$, $y(0) = y_0$. It has to be shown that $y = y_0 e^{kx}$.

Define $v = y(x)e^{-kx}$. This defines a change of variable from y into v. Then

$v' = (e^{-kx}y)'$	Compute v' from $v = e^{-kx}y$.
$= -ke^{-kx}y + e^{-kx}y'$	Apply the product rule $(uy)' = u'y + uy'$.
$= -ke^{-kx}y + e^{-kx}(ky)$	Use the differential equation $y' = ky$.
$= 0.$	The terms cancel.

In summary, $v' = 0$ for all x. The calculus result to be applied is:

> The only function $v(x)$ that satisfies $v'(x) = 0$ on an interval is $v(x) =$ constant.

The conclusion is $v(x) = v_0$ for some constant v_0. Then $v = e^{-kx}y$ gives $y = v_0 e^{kx}$. Setting $x = 0$ implies $v_0 = y_0$ and finally $y = y_0 e^{kx}$. This completes the verification.

Newton Cooling Solution Verification (Theorem 1). The substitution $A(t) = u(t) - u_1$ will be applied to find an equivalent growth-decay equation:

$\dfrac{dA}{dt} = \dfrac{d}{dt}\left(u(t) - u_1\right)$	Definition of $A = u - u_1$.
$= u'(t) - 0$	Derivative rules applied.
$= -h(u(t) - u_1)$	Cooling differential equation applied.
$= -hA(t)$	Definition of A.

The conclusion is that $A'(t) = -hA(t)$. Then $A(t) = A_0 e^{-ht}$, from the theory of growth-decay equations. The substitution gives $u(t) - u_1 = A_0 e^{-ht}$, which is equivalent to equation (5), provided $A_0 = u_0 - u_1$. The proof is complete.

[1] The German mathematician E. E. Kümmer, in his paper in 1834, republished in 1887 in *J. für die reine und angewandte Math.*, considered changes of variable $y = wv$, where w is a given function of x and v is the new variable that replaces y.

Logistic Solution Verification (Theorem 2). Given $a > 0$, $b > 0$ and the logistic equation $P' = (a - bP)P$, the plan is to derive the solution formula

$$P(t) = \frac{aP(0)e^{at}}{bP(0)e^{at} + a - bP(0)}.$$

Assume $P(t)$ satisfies the logistic equation. Suppose it has been shown (see below) that the variable $u = P/(a - bP)$ satisfies $u' = au$. By the exponential theory, $u = u_0 e^{at}$, hence

$$\begin{aligned}
P &= \frac{au}{1 + bu} && \text{Solve } u = P/(a-bP) \text{ for } P \text{ in terms of } u. \\
&= \frac{au_0 e^{at}}{1 + bu_0 e^{at}} && \text{Substitute } u = u_0 e^{at}. \\
&= \frac{ae^{at}}{1/u_0 + be^{at}} && \text{Divide by } u_0. \\
&= \frac{ae^{at}}{(a - bP(0))/P(0) + be^{at}} && \text{Use } u_0 = u(0) \text{ and } u = P/(a-bP). \\
&= \frac{aP(0)e^{at}}{bP(0)e^{at} + a - bP(0)}. && \text{Formula verified.}
\end{aligned}$$

The derivation using the substitution $u = P/(a - bP)$ requires only differential calculus. The substitution was found by afterthought, already knowing the solution; historically, integration methods have been applied.

The change of variables $(t, P) \to (t, u)$ is used to justify the relation $u' = au$ as follows.

$$\begin{aligned}
u' &= \left(\frac{P}{a - bP}\right)' && \text{It will be shown that } u' = au. \\
&= \frac{P'(a - bP) - P(-bP')}{(a - bP)^2} && \text{Quotient rule applied.} \\
&= \frac{a(a - bP)P}{(a - bP)^2} && \text{Simplify and substitute the equation } P' = (a-bP)P. \\
&= au && \text{Substitute } u = P/(a-bP).
\end{aligned}$$

This completes the motivation for the formula. To verify that it works in the differential equation is a separate issue, which is settled in the exercises.

Exercises 1.1

Growth-decay Recipe. Solve the given initial value problem using the growth-decay recipe; see page 3 and Example 1, page 7.

1. $y' = -3y$, $y(0) = 20$

2. $y' = 3y$, $y(0) = 1$

3. $3A' = A$, $A(0) = 1$

4. $4A' + A = 0$, $A(0) = 3$

5. $3P' - P = 0$, $P(0) = 10$

6. $4P' + 3P = 0$, $P(0) = 11$

7. $I' = 0.005I$, $I(t_0) = I_0$

8. $I' = -0.015I$, $I(t_0) = I_0$

9. $y' = \alpha y$, $y(t_0) = 1$

10. $y' = -\alpha y$, $y(t_0) = y_0$

Growth-decay Theory.

11. Graph without a computer $y = 10(2^x)$ on $-3 \le x \le 3$.

12. Graph without a computer $y = 10(2^{-x})$ on $-3 \le x \le 3$.

13. Find the doubling time for the growth model $P = 100e^{0.015t}$.

14. Find the doubling time for the growth model $P = 1000e^{0.0195t}$.

15. Find the elapsed time for the decay model $A = 1000e^{-0.11237t}$ until $|A(t)| < 0.00001$.

16. Find the elapsed time for the decay model $A = 5000e^{-0.01247t}$ until $|A(t)| < 0.00005$.

Newton Cooling Recipe.

Solve the given cooling model. Follow Example 2 on page 7.

17. $u' = -10(u-4)$, $u(0) = 5$

18. $y' = -5(y-2)$, $y(0) = 10$

19. $u' = 1 + u$, $u(0) = 100$

20. $y' = -1 - 2y$, $y(0) = 4$

21. $u' = -10 + 4u$, $u(0) = 10$

22. $y' = 10 + 3y$, $y(0) = 1$

23. $2u' + 3 = 6u$, $u(0) = 8$

24. $4y' + y = 10$, $y(0) = 5$

25. $u' + 3(u+1) = 0$, $u(0) = -2$

26. $u' + 5(u+2) = 0$, $u(0) = -1$

27. $\alpha' = -2(\alpha - 3)$, $\alpha(0) = 10$

28. $\alpha' = -3(\alpha - 4)$, $\alpha(0) = 12$

Newton Cooling Model.

The cooling model $u(t) = u_0 + A_0 e^{-ht}$ is applied; see page 4. Methods parallel those in the flask cooling example, page 8, and the baking example, page 9.

29. (Ingot Cooling) A metal ingot cools in the air at temperature 20C from 130C to 75C in one hour. Predict the cooling time to 23C.

30. (Rod Cooling) A plastic rod cools in a large vat of 12-degree Celsius water from 75C to 20C in 4 minutes. Predict the cooling time to 15C.

31. (Murder Mystery) A body discovered at 1:00 in the afternoon, March 1, 1929, had temperature 80F. Over the next hour the body's temperature dropped to 76F. Estimate the date and time of the murder.

32. (Time of Death) A dead person found in a 40F river had body temperature 70F. The coroner requested that the body be left in the river for 45 minutes, whereupon the body's temperature was 63F. Estimate the time of death, relative to the discovery of the body.

Verhulst Recipe.

Solve the given Verhulst logistic equation using formula (8). Follow Example 3 on page 7.

33. $P' = P(2-P)$, $P(0) = 1$

34. $P' = P(4-P)$, $P(0) = 5$

35. $y' = y(y-1)$, $y(0) = 2$

36. $y' = y(y-2)$, $y(0) = 1$

37. $A' = A - 2A^2$, $A(0) = 3$

38. $A' = 2A - 5A^2$, $A(0) = 1$

39. $F' = 2F(3-F)$, $F(0) = 2$

40. $F' = 3F(2-F)$, $F(0) = 1$

Inverse Modeling. Given the model, find the differential equation and initial condition.

41. $A = A_0 e^{4t}$

42. $A = A_0 e^{-3t}$

43. $P = 1000e^{-0.115t}$

44. $P = 2000e^{-7t/5}$

45. $u = 1 + e^{-3t}$

46. $u = 10 - 2e^{-2t}$

47. $P = \dfrac{10}{10 - 8e^{-2t}}$

48. $P = \dfrac{5}{15 - 14e^{-t}}$

49. $P = \dfrac{1}{5 - 4e^{-t}}$

50. $P = \dfrac{2}{4 - 3e^{-t}}$

Populations. The following exercises use Malthus's population theory, page 5, and the Malthusian model $P(t) = P_0 e^{kt}$. Methods appear in Examples 4 and 5; see page 7.

51. **(World Population)** In June of 1993, the world population of $5,500,000,000$ people was increasing at a rate of $250,000$ people per day. Predict the date when the population reaches 10 billion.

52. **(World Population)** Suppose the world population at time $t = 0$ is 5 billion. How many years before that was the population one billion?

53. **(Population Doubling)** A population of rabbits increases by 10% per year. In how many years does the population double?

54. **(Population Tripling)** A population of bacteria increases by 15% per day. In how many days does the population triple?

55. **(Population Growth)** Trout in a river are increasing by 15% in 5 years. To what population size does 500 trout grow in 15 years?

56. **(Population Growth)** A region of 400 acres contains 1000 forest mushrooms per acre. The population is decreasing by 150 mushrooms per acre every 2 years. Find the population size for the 400-acre region in 15 years.

Verhulst Equation. Write out the solution to the given differential equation and, when it makes sense, report the carrying capacity

$$M = \lim_{t\to\infty} P(t).$$

57. $P' = (1 - P)P$

58. $P' = (2 - P)P$

59. $P' = 0.1(3 - 2P)P$

60. $P' = 0.1(4 - 3P)P$

61. $P' = 0.1(3 + 2P)P$

62. $P' = 0.1(4 + 3P)P$

63. $P' = 0.2(5 - 4P)P$

64. $P' = 0.2(6 - 5P)P$

65. $P' = 11P - 17P^2$

66. $P' = 51P - 13P^2$

Logistic Equation. The following exercises use Verhulst's logistic equation $P' = (a - bP)P$, page 6. Some methods appear on page 11.

67. **(Protozoa)** Experiments on the protozoa *Paramecium* determined growth rate $a = 2.309$ and carrying capacity $a/b = 375$ using initial population $P(0) = 5$. Establish the formula $P(t) = \dfrac{375}{1 + 74e^{-2.309t}}$.

68. **(World Population)** Demographers incorrectly projected the world population in the year 2000 as 6.5 billion (in 1970) and 5.9 billion (in 1976). Use $P(1965) = 3.358 \times 10^9$, $a = 0.029$ and carrying capacity $a/b = 1.0760668 \times 10^{10}$ to compute the logistic equation projection for year 2000.

69. **(Harvesting)** A fish population satisfying $P' = (a - bP)P$ is subjected to harvesting, the new model being $P' = (a-bP)P - H$. Assume $a = 0.04$, $a/b = 5000$ and $H = 1000$. Using algebra, rewrite it as $P' = b(\alpha - P)(P - \beta)$ in terms of the roots α, β of $ay - by^2 - H = 0$. Apply the change of variables $u = (\alpha - P)/(P - \beta)$ to solve it.

70. **(Extinction)** Let an endangered species satisfy $P' = bP^2 - aP$ for $a > 0$, $b > 0$. The term bP^2 represents births due to chance encounters of males and females, while the term aP represents deaths. Use the change of variable $u = P/(bP - a)$ to solve it. Show from the answer that population sizes below a/b become extinct.

71. **(Logistic Solution)** Let $P = au/(1 + bu)$, $u = u_0 e^{at}$, $u_0 = P_0/(a - bP_0)$. Verify that $P(t)$ is a solution the differential equation $P' = (a - bP)P$ and $P(0) = P_0$.

72. **(Logistic Equation)** Let k, α, β be positive constants, $\alpha < \beta$. Solve $w' = k(\alpha - w)(\beta - w)$, $w(0) = w_0$ by the substitution $u = (\alpha - w)/(\beta - w)$, showing that $w = (\alpha - \beta u)/(1 - u)$, $u = u_0 e^{(\alpha - \beta)kt}$, $u_0 = (\alpha - w_0)/(\beta - w_0)$. This equation is a special case of the harvesting equation $P' = (a - bP)P + H$.

Growth-Decay Uniqueness Proof.

73. State precisely and give a calculus text reference for *Rolle's Theorem*, which says that a function vanishing at $x = a$ and $x = b$ must have slope zero at some point in $a < x < b$.

74. Apply Rolle's Theorem to prove that a differentiable function $v(x)$ with $v'(x) = 0$ on $a < x < b$ must be constant.

1.2 Exponential Application Library

The model differential equation $y' = ky$, and its variants via a change of variables, appears in various applications to biology, chemistry, finance, science and engineering. All the applications below use the exponential model $y = y_0 e^{kt}$.

Light Intensity	Chemical Reactions
Electric Circuits	Drug Elimination
Drug Dosage	Continuous Interest
Radioactive Decay	Radiocarbon Dating

Light Intensity

Physics defines the **lumen unit** to be the light flux through a solid unit angle from a point source of 1/621 watts of yellow light.[2] The lumen is designed for measuring **brightness** , as perceived by the human eye. The **intensity** $E = \frac{F}{A}$ is the flux F per unit area A, with units Lux or Foot-candles (use $A = 1\text{m}^2$ or $A = 1\text{ft}^2$, respectively). At a radial distance r from a point source, in which case $A = 4\pi r^2$, the intensity is given by the **inverse square law**

$$E = \frac{F}{4\pi r^2}.$$

An **exposure meter** , which measures incident or reflected light intensity, consists of a body, a photocell and a readout in units of Lux or Foot-candles. Light falling on the photocell has energy, which is transferred by the photocell into electrical current and ultimately converted to the readout scale.

In classical physics experiments, a jeweler's bench is illuminated by a source of 8000 lumens. The experiment verifies the inverse square law, by reading an exposure meter at 1/2, 1 and 3/2 meters distance from the source.

As a variant on this experiment, consider a beaker of jeweler's cleaning fluid which is placed over the exposure meter photocell; see Figure 3. Successive meter readings with beaker depths of 0, 5, 10, 15 centimeters show that fluid **absorption** significantly affects the meter readings. Photons[3] striking the fluid convert into heat, which accounts for the rapid loss of intensity at depth in the fluid.

[2]Precisely, the wavelength of the light is 550-nm. The unit is equivalent to one **candela** , one of the seven basic SI units, which is the luminous intensity of one sixtieth of a square centimeter of pure platinum held at 1770C.

[3]A photon is the quantum of electromagnetic radiation, of energy $h\nu$, where ν is the radiation frequency and h is Planck's constant.

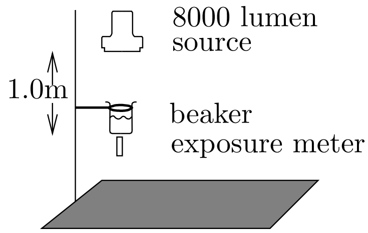


Figure 3. Jeweler's bench experiment.
The exposure meter measures light intensity at the beaker's base.

Empirical evidence from experiments suggests that light intensity $I(x)$ at a depth x in the fluid *changes at a rate proportional to itself*, that is,

$$\frac{dI}{dx} = -kI. \tag{9}$$

If I_0 is the surface intensity and $I(x)$ is the intensity at depth x meters, then the theory of growth-decay equations applied to (9) gives the solution

$$I(x) = I_0 e^{-kx}. \tag{10}$$

Equation (10) says that the intensity $I(x)$ at depth x is a percentage of the surface intensity I_0, the percentage decreasing with depth x.

Electric Circuits

Classical physics analyzes the RC-circuit in Figure 4 and the LR-circuit in Figure 5. The physics background will be reviewed.

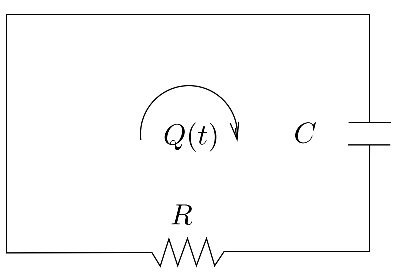


Figure 4. An RC-Circuit, no emf.

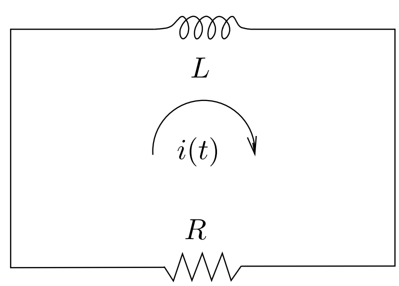


Figure 5. An LR-Circuit, no emf.

First, the **charge** $Q(t)$ in coulombs and the **current** $I(t)$ in amperes are related by the rate formula $I(t) = Q'(t)$. Secondly, there are some empirical laws that are used. There is **Kirchhoff's voltage law** :

> The algebraic sum of the voltage drops around a closed loop is zero.

Kirchhoff's **node law** is not used here, because only one loop appears in the examples.

There are the **voltage drop formulas** for an inductor of L henrys, a resistor of R ohms and a capacitor of C farads:

Faraday's law	$V_L = LI'$
Ohm's law	$V_R = RI$
Coulomb's law	$V_C = Q/C$

In Figure 4, Kirchhoff's law implies $V_R + V_C = 0$. The voltage drop formulas show that the charge $Q(t)$ satisfies $RQ'(t) + (1/C)Q(t) = 0$. Let $Q(0) = Q_0$. Growth-decay theory, page 3, gives $Q(t) = Q_0 e^{-t/(RC)}$.

In Figure 5, Kirchhoff's law implies that $V_L + V_R = 0$. By the voltage drop formulas, $LI'(t) + RI(t) = 0$. Let $I(0) = I_0$. Growth-decay theory gives $I(t) = I_0 e^{-Rt/L}$.

In summary:

RC-**Circuit** $Q = Q_0 e^{-t/(RC)}$	$RQ' + (1/C)Q = 0, \quad Q(0) = Q_0,$
LR-**Circuit** $I = I_0 e^{-Rt/L}$	$LI' + RI = 0, \quad I(0) = I_0.$

The ideas outlined here are illustrated in Examples 9 and 10, page 21.

Interest

The notion of **simple interest** is based upon the financial formula

$$A = (1 + r)^t A_0$$

where A_0 is the initial amount, A is the final amount, t is the number of years and r is the **annual interest rate** or **rate per annum** (5% means $r = 5/100$). The **compound interest** formula is

$$A = \left(1 + \frac{r}{n}\right)^{nt} A_0$$

where n is the number of times to compound interest per annum. Use $n = 4$ for **quarterly interest** and $n = 360$ for **daily interest** .

The topic of **continuous interest** has its origins in taking the limit as $n \to \infty$ in the compound interest formula. The answer to the limit problem is the **continuous interest formula**

$$A = A_0 e^{rt}$$

which by the growth-decay theory arises from the initial value problem

$$\begin{cases} A'(t) = rA(t), \\ A(0) = A_0. \end{cases}$$

Shown on page 26 are the details for taking the limit as $n \to \infty$ in the compound interest formula. In analogy with population theory, the following statement can be made about continuous interest.

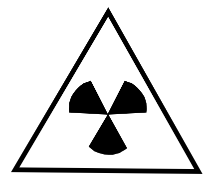
> The amount accumulated by continuous interest increases at a rate proportional to itself.

Applied often in interest calculations is the **geometric sum formula** from algebra:

$$1 + r + \cdots + r^n = \frac{r^{n+1} - 1}{r - 1}.$$

Radioactive Decay

A constant fraction of the atoms present in a radioactive isotope will spontaneously decay into another isotope of the identical element or else into atoms of another element. Empirical evidence gives the following decay law:



> A radioactive isotope decays at a rate proportional to the amount present.

In analogy with population models the differential equation for radioactive decay is

$$\frac{dA}{dt} = -kA(t),$$

where $k > 0$ is a physical constant called the **decay constant** , $A(t)$ is the number of atoms of radioactive isotope and t is measured in years.

Radiocarbon Dating. The decay constant $k \approx 0.0001245$ is known for carbon-14 (^{14}C). The model applies to measure the date that an organism died, assuming it metabolized atmospheric carbon-14.

The idea of radiocarbon dating is due to Willard S. Libby[4] in the late 1940s. The basis of the chemistry is that radioactive carbon-14, which has two more electrons than stable carbon-12, gives up an electron to become stable nitrogen-14. Replenishment of carbon-14 by cosmic rays keeps atmospheric carbon-14 at a nearly constant ratio with ordinary carbon-12 (this was Libby's assumption). After death, the radioactive decay of carbon-14 depletes the isotope in the organism. The percentage of depletion from atmospheric levels of carbon-14 gives a measurement that dates the organism.

Definition 2 (Half-Life)
The **half-life** of a radioactive isotope is the time T required for half of the isotope to decay. In functional notation, it means $A(T) = A(0)/2$, where $A(t) = A(0)e^{kt}$ is the amount of isotope at time t.

[4]Libby received the Nobel Prize for Chemistry in 1960.

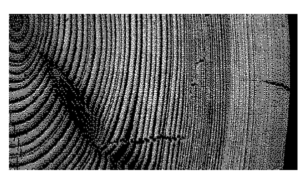
For carbon-14, the half-life is 5568 years plus or minus 30 years, according to Libby (some texts and references give 5730 years). The decay constant $k \approx 0.0001245$ for carbon-14 arises by solving for $k = \ln(2)/5568$ in the equation $A(5568) = \frac{1}{2}A(0)$. Experts believe that carbon-14 dating methods tend to underestimate the age of a fossil.

Uranium-238 undergoes decay via alpha and beta radiation into various nuclides, the half-lives of which are shown in Table 1. The table illustrates the range of possible half-lives for a radioactive substance.

Table 1. Uranium-238 nuclides by alpha or beta radiation.

Nuclide	**Half-Life**
uranium-238	4,500,000,000 years
thorium-234	24.5 days
protactinium-234	1.14 minutes
uranium-234	233,000 years
thorium-230	83,000 years
radium-236	1,590 years
radon-222	3.825 days
polonium-218	3.05 minutes
lead-214	26.8 minutes
bismuth-214	19.7 minutes
polonium-214	0.00015 seconds
lead-210	22 years
bismuth-210	5 days
polonium-210	140 days
lead-206	stable

Tree Rings. Libby's work was based upon calculations from sequoia tree rings. Later investigations of 4000-year old trees showed that carbon ratios have been non-constant over past centuries.



Libby's method is advertised to be useful for material 200 years to $40,000$ years old. Older material has been dated using the ratio of disintegration byproducts of potassium-40, specifically argon-40 to calcium-40.

An excellent reference for dating methods, plus applications and historical notes on the subject, is Chapter 1 of Braun [**?**].

Chemical Reactions

If the molecules of a substance decompose into smaller molecules, then an empirical law of **first-order reactions** says that the decomposition rate is proportional to the amount of substance present. In mathematical notation, this means

$$\frac{dA}{dt} = -hA(t)$$

where $A(t)$ is the amount of the substance present at time t and h is a physical constant called the **reaction constant** .

The **law of mass action** is used in chemical kinetics to describe **second-order reactions** . The law describes the amount $X(t)$ of chemical C produced by the combination of two chemicals A and B. A chemical derivation produces a rate equation

$$X' = k(\alpha - X)(\beta - X), \quad X(0) = X_0, \tag{11}$$

where k, α and β are physical constants, $\alpha < \beta$; see Zill-Cullen [Z-C], Chapter 2. The substitution $u = (\alpha - X)/(\beta - X)$ is known to transform (11) into $u' = k(\alpha - \beta)u$ (see page 11 for the technique and the exercises in this section). Therefore,

$$X(t) = \frac{\alpha - \beta u(t)}{1 - u(t)}, \quad u(t) = u_0 e^{(\alpha-\beta)kt}, \quad u_0 = \frac{\alpha - X_0}{\beta - X_0}. \tag{12}$$

Drug Elimination

Some drugs are eliminated from the bloodstream by an animal's body in a predictable fashion. The amount $D(t)$ in the bloodstream declines at a rate proportional to the amount already present. Modeling drug elimination exactly parallels radioactive decay, in that the translated mathematical model is

$$\frac{dD}{dt} = -hD(t),$$

where $h > 0$ is a physical constant, called the **elimination constant** of the drug.

Oral drugs must move through the digestive system and into the gut before reaching the bloodstream. The model $D'(t) = -hD(t)$ applies only after the drug has reached a stable concentration in the bloodstream and the body begins to eliminate the drug.

Examples

8 **Example (Light intensity)** Light intensity in a lake is decreased by 75% at depth one meter. At what depth is the intensity decreased by 95%?

Solution: The answer is 2.16 meters (7 feet, $1\frac{1}{16}$ inches). This depth will be justified by applying the light intensity model $I(x) = I_0 e^{-kx}$, where I_0 is the surface intensity.

At one meter the intensity is $I(1) = I_0 e^{-k}$, but also it is given as $0.25 I_0$. The equation $e^{-k} = 0.25$ results, to determine $k = \ln 4 \approx 1.3862944$. To find the depth x when the intensity has decreased by 95%, solve $I(x) = 0.05 I_0$ for x. The value I_0 cancels from this equation, leaving $e^{-kx} = 1/20$. The usual logarithm methods give $x \approx 2.2$ meters, as follows:

$\ln e^{-kx} = \ln(1/20)$	Take the logarithm across $e^{-kx} = 1/20$.
$-kx = -\ln(20)$	Use $\ln e^u = u$ and $-\ln u = \ln(1/u)$.
$x = \dfrac{\ln(20)}{k}$	Divide by $-k$.
$= \dfrac{\ln(20)}{\ln(4)}$	Use $k = \ln(4)$.
≈ 2.16 meters.	Only 5% of the surface intensity remains at 2.16 meters.

9 Example (RC-Circuit) Solve the RC-circuit equation $RQ' + (1/C)Q = 0$ when $R = 2$, $C = 10^{-2}$ and the voltage drop across the capacitor at $t = 0$ is 1.5 volts.

Solution: The solution is $Q = 0.015e^{-50t}$. To justify this equation, start with the voltage drop formula $V_C = Q/C$, page 17. Then $1.5 = Q(0)/C$ implies $Q(0) = 0.015$. The differential equation is $Q' + 50Q = 0$; page 3 gives the solution $Q = Q(0)e^{-50t}$.

10 Example (LR-Circuit) Solve the LR-circuit equation $LI' + RI = 0$ when $R = 2$, $L = 0.1$ and the voltage drop across the resistor at $t = 0$ is 1.0 volts.

Solution: The solution is $I = 0.5e^{-20t}$. To justify this equation, start with the voltage drop formula $V_R = RI$, page 17. Then $1.0 = RI(0)$ implies $I(0) = 0.5$. The differential equation is $I' + 20I = 0$; page 3 gives the solution $I = I(0)e^{-20t}$.

11 Example (Compound Interest) Compute the fixed monthly payment for a 5-year auto loan of \$18,000 at 9% per annum, using (a) daily interest and (b) continuous interest.

Solution: The payments are (a) \$373.94 and (b) \$373.95, which differ by one cent; details below.

Let $A_0 = 18000$ be the initial amount. It will be assumed that the first payment is due after 30 days and monthly thereafter. To simplify the calculation, a **day** is defined to be 1/360th of a year, regardless of the number of days in that year, and payments are applied every 30 days. Late fees apply if the payment is not received within the **grace period** , but it will be assumed here that all payments are made on time.

Part (a). The **daily interest rate** is $R = 0.09/360$ applied for 1800 periods (5 years). Between payments P, daily interest is applied to the balance $A(t)$ owed after t periods. The balance grows between payments and then decreases on the day of the payment. The problem is to *find P so that* $A(1800) = 0$.

Payments are subtracted every 30 periods making balance $A(30k)$. Let $B = (1+R)^{30}$ and $A_k = A(30k)$. Then

$A_k = A(30k)$	Balance after k months.
$\quad = A_0B^k - P(1 + \cdots + B^{k-1})$	For $k = 1, 2, 3, \ldots$
$\quad = A_0B^k - P\dfrac{B^k - 1}{B - 1}$	Geometric sum formula applied, page 18.
$A_0B^{60} = P\dfrac{B^{60} - 1}{B - 1}$	Use $A(1800) = 0$, which corresponds to $k = 60$.
$P = A_0(B - 1)\dfrac{B^{60}}{B^{60} - 1}$	Solve for P.
$\quad = 373.93857$	By calculator.

Part (b). The details are the same except for the method of applying interest. Let $s = 30(0.09)/360$, then

$A_k = A_0e^{ks} - Pe^{ks-s}\left(1 + e^{-s} + \cdots + e^{-ks+s}\right)$	For $k = 1, 2, 3, \ldots$, by examination of cases $A(30)$ and $A(60)$.
$\quad = A_0e^{ks} - Pe^{ks-s}\left(\dfrac{e^{-ks} - 1}{e^{-s} - 1}\right)$	Apply the geometric sum formula with common ratio e^{-s}.
$A_0e^{60s} = Pe^{60s-s}\dfrac{e^{-60s} - 1}{e^{-s} - 1}$	Set $k = 60$ and $A(1800) = 0$ in the previous formula.
$P = A_0\dfrac{-e^s + 1}{e^{-60s} - 1}$	Solve for P.
$\quad = 373.94604$	By calculator.

12 **Example (Effective Annual Yield)** A bank advertises an effective annual yield of 5.73% for a certificate of deposit with continuous interest rate 5.5% per annum. Justify the rate.

Solution: The **effective annual yield** is the simple annual interest rate which gives the same account balance after one year. The issue is whether one year means 365 days or 360 days, since banks do business on a 360-day cycle.

Suppose first that one year means 365 days. The model used for a saving account is $A(t) = A_0e^{rt}$ where $r = 0.055$ is the interest rate per annum. For one year, $A(1) = A_0e^r$. Then $e^r = 1.0565406$, that is, the account has increased in one year by 5.65%. The *effective annual yield* is 0.0565 or 5.65%.

Suppose next that one year means 360 days. Then the bank pays 5.65% for only 360 days to produce a balance of $A_1 = A_0e^r$. The extra 5 days make 5/360 years, therefore the bank records a balance of $A_1e^{5r/360}$ which is $A_0e^{365r/360}$. The rate for 365 days is then 5.73%, by the calculation

$$\frac{365}{360}0.0565406 = 0.057325886.$$

13 Example (Retirement Funds) An engineering firm offers a starting salary of 40 thousand per year, which is expected to increase 3% per year. Retirement contributions are 11% of salary, deposited monthly, growing at 6% continuous interest per annum. The company advertises a million dollars in retirement funds after 40 years. Justify the claim.

Solution: The salary is estimated to be $S(t) = 40000(1.03)^t$ after t years, because it starts with $S(0) = 40000$ and each year it takes a 3% increment. After 39 years of increases the salary has increased from \$40,000 to \$126,681.

Let A_n be the amount in the retirement account at the end of year n. Let $P_n = (40000(1.03)^n)(0.11)/12$ be the monthly salary for year $n+1$. The interest rates are $r = 0.06$ (annual) and $s = 0.06/12$ (monthly). For brevity, let $R = 1.03$.

During the first year, the retirement account accumulates 12 times for a total

$A_1 = P_0 + \cdots + P_0 e^{11s}$	Continuous interest at rate s on amount P_0 for 1 through 11 months.
$= P_0 \dfrac{e^r - 1}{e^s - 1}$	Geometric sum with common ratio e^s.
$= 4523.3529.$	Retirement balance after one year.

During the second and later years the retirement account accumulates by the rule

$A_{n+1} = A_n e^r + P_n + P_n e^s + \cdots + P_n e^{11s}$	One year's accumulation at continuous rate r on amount A_n plus monthly accumulations on retirement contributions P_n.
$= A_n e^r + P_n \dfrac{e^r - 1}{e^s - 1}$	Apply the geometric sum formula with common ratio e^s.
$= A_n e^r + R^n P_0 \dfrac{e^r - 1}{e^s - 1}$	Use $P_n = P_0 R^n$.
$= A_n e^r + R^n A_1.$	Apply $A_1 = P_0 \dfrac{e^r - 1}{e^s - 1}$.

After examining cases $n = 1, 2, 3$, the recursion is solved to give

$$A_n = A_1 \sum_{k=0}^{n-1} e^{kr} R^{n-1-k}.$$

To establish this formula, induction is applied:

$A_{n+1} = A_n e^r + R^n A_1$	Derived above.
$= A_1 e^r \sum_{k=0}^{n-1} e^{kr} R^{n-1-k} + R^n A_1$	Apply the induction hypothesis.
$= A_1 \sum_{k=0}^{n} e^{kr} R^{n-k}$	Rewrite the sum indices.
$= A_1 R^n \dfrac{(e^r/R)^{n+1} - 1}{e^r/R - 1}.$	Use the geometric sum formula with common ratio e^r/R.

The advertised retirement fund after 40 years should be the amount A_{40}, which is obtained by setting $n = 39$ in the last equality. Then $A_{40} = 1102706.60$.

14 Example (Half-life of Radium) A radium sample loses $1/2$ percent due to disintegration in 12 years. Verify the half-life of the sample is about $1,660$ years.

Solution: The decay model $A(t) = A_0 e^{-kt}$ applies. The given information $A(12) = 0.995A(0)$ reduces to the exponential equation $e^{-12k} = 0.995$ with solution $k = \ln(1000/995)/12$. The half-life T satisfies $A(T) = \frac{1}{2}A(0)$, which reduces to $e^{-kT} = 1/2$. Since k is known, the value T can be found as $T = \ln(2)/k \approx 1659.3909$ years.

15 Example (Radium Disintegration) The disintegration reaction

$$ {}_{88}R^{226} \longrightarrow {}_{88}R^{224} $$

of radium-226 into radon has a half-life of 1700 years. Compute the decay constant k in the decay model $A' = -kA$.

Solution: The half-life equation is $A(1700) = \frac{1}{2}A(0)$. Since $A(t) = A_0 e^{-kt}$, the equation reduces to $e^{-1700k} = 1/2$. The latter is solved for k by logarithm methods (see page 7), giving $k = \ln(2)/1700 = 0.00040773364$.

16 Example (Radiocarbon Dating) The ratio of carbon-14 to carbon-12 in a dinosaur fossil is 6.34 percent of the current atmospheric ratio. Verify the dinosaur's death was about $22,160$ years ago.

Solution: The method due to Willard Libby will be applied, using his assumption that the ratio of carbon-14 to carbon-12 in living animals is equal to the atmospheric ratio. Then carbon-14 depletion in the fossil satisfies the decay law $A(t) = A_0 e^{-kt}$ for some parameter values k and A_0.

Assume the half-life of carbon-14 is 5568 years. Then $A(5568) = \frac{1}{2}A(0)$ (see page 18). This equation reduces to $A_0 e^{-5568k} = \frac{1}{2}A_0 e^0$ or $k = \ln(2)/5568$. In short, k is known but A_0 is unknown. It is not necessary to determine A_0 in order to do the verification.

At the time t_0 in the past when the organism died, the amount A_1 of carbon-14 began to decay, reaching the value $6.34A_1/100$ at time $t = 0$ (the present). Therefore, $A_0 = 0.0634A_1$ and $A(t_0) = A_1$. Taking this last equation as the starting point, the final calculation proceeds as follows.

$A_1 = A(t_0)$	The amount of carbon-14 at death is A_1, $-t_0$ years ago.
$\quad = A_0 e^{-kt_0}$	Apply the decay model $A = A_0 e^{-kt}$ at $t = t_0$.
$\quad = 0.0634A_1 e^{-kt_0}$	Use $A_0 = 6.34A_1/100$.

The value A_1 cancels to give the new relation $1 = 0.0634e^{-kt_0}$. The value $k = \ln(2)/5568$ gives an exponential equation to solve for t_0:

$e^{kt_0} = 0.0634$	Multiply by e^{kt_0} to isolate the exponential.
$\ln e^{kt_0} = \ln(0.0634)$	Take the logarithm of both sides.
$t_0 = \dfrac{1}{k}\ln(0.0634)$	Apply $\ln e^u = u$ and divide by k.
$= \dfrac{5568}{\ln 2}\ln(0.0634)$	Substitute $k = \ln(2)/5568$.
$= -22157.151$ years.	By calculator. The fossil's age is the negative.

17 Example (Percentage of an Isotope) A radioactive isotope disintegrates by 5% in ten years. By what percentage does it disintegrate in one hundred years?

Solution: The answer is not 50%, as is widely reported by lay persons. The correct answer is 40.13%. It remains to justify this non-intuitive answer.

The model for decay is $A(t) = A_0e^{-kt}$. The decay constant k is known because of the information *... disintegrates by 5% in ten years.* Translation to equations produces $A(10) = 0.95A_0$, which reduces to $e^{-10k} = 0.95$. Solving with logarithms gives $k = 0.1\ln(100/95) \approx 0.0051293294$.

After one hundred years, the isotope present is $A(100)$, and the percentage is $100\frac{A(100)}{A(0)}$. The common factor A_0 cancels to give the percentage $100e^{-100k} \approx 59.87$. The reduction is 40.13%.

To reconcile the lay person's answer, observe that the amounts present after one, two and three years are $0.95A_0$, $(0.95)^2A_0$, $(0.95)^3A_0$. The lay person should have guessed 100 times $1 - (0.95)^{10}$, which is 40.126306. The common error is to simply multiply 5% by the ten periods of ten years each. By this erroneous reasoning, the isotope would be depleted in two hundred years, whereas the decay model says that about 36% of the isotope remains!

18 Example (Chemical Reaction) The manufacture of t-butyl alcohol from t-butyl chloride is made by the chemical reaction

$$(CH_3)_3CCL + NaOH \longrightarrow (CH_3)_3COH + NaCL.$$

Model the production of t-butyl alcohol, when N% of the chloride remains after t_0 minutes.

Solution: It will be justified that the model for alcohol production is $A(t) = C_0(1 - e^{-kt})$ where $k = \ln(100/N)/t_0$, C_0 is the initial amount of chloride and t is in minutes.

According to the theory of first-order reactions, the model for chloride depletion is $C(t) = C_0e^{-kt}$ where C_0 is the initial amount of chloride and k is the reaction constant. The alcohol production is $A(t) = C_0 - C(t)$ or $A(t) = C_0(1 - e^{-kt})$. The reaction constant k is found from the initial data $C(t_0) = \frac{N}{100}C_0$, which results in the exponential equation $e^{-kt_0} = N/100$. Solving the exponential equation gives $k = \ln(100/N)/t_0$.

19 **Example (Drug Dosage)** A veterinarian applies general anesthesia to animals by injection of a drug into the bloodstream. Predict the drug dosage to anesthetize a 25-pound animal for thirty minutes, given:

1. The drug requires an injection of 20 milligrams per pound of body weight in order to work.
2. The drug eliminates from the bloodstream at a rate proportional to the amount present, with a half-life of 5 hours.

Solution: The answer is about 536 milligrams of the drug. This amount will be justified using exponential modeling.

The drug model is $D(t) = D_0 e^{-ht}$, where D_0 is the initial dosage and h is the elimination constant. The half-life information $D(5) = \frac{1}{2}D_0$ determines $h = \ln(2)/5$. Depletion of the drug in the bloodstream means the drug levels are always decreasing, so it is enough to require that the level at 30 minutes exceeds 20 times the body weight in pounds, that is, $D(1/2) > (20)(25)$. The critical value of the initial dosage D_0 then occurs when $D(1/2) = 500$ or $D_0 = 500e^{h/2} = 500e^{0.1\ln(2)}$, which by calculator is approximately 535.88673 milligrams.

Drugs like sodium pentobarbital behave somewhat like this example, although injection in a single dose may not be preferable. An intravenous drip can be employed to sustain the blood levels of the drug, keeping the level closer to the target 500 milligrams.

Details and Proofs

Verification of Continuous Interest by Limiting. Derived here is the continuous interest formula by limiting as $n \to \infty$ in the compound interest formula.

$\left(1+\dfrac{r}{n}\right)^{nt} = B^{nt}$	In the exponential rule $B^x = e^{x\ln B}$, the base is $B = 1 + r/n$.
$= e^{nt\ln B}$	Use $B^x = e^{x\ln B}$ with $x = nt$.
$= e^{\frac{r\ln(1+u)}{u}t}$	Substitute $u = r/n$. Then $u \to 0$ as $n \to \infty$.
$\approx e^{rt}$	Because $\ln(1+u)/u \approx 1$ as $u \to 0$, by L'Hospital's rule.

Exercises 1.2

Light Intensity. The following exercises apply the theory of light intensity on page 15, using the model $I(t) = I_0 e^{-kx}$ with x in meters. Methods parallel Example 8 on page 20.

1. The light intensity is $I(t) = I_0 e^{-1.4x}$ in a certain swimming pool. At what depth does the light intensity fall off by 50%?

2. The light intensity in a swimming pool falls off by 50% at a depth of 2.5 meters. Find the depletion constant k in the exponential model.

3. Plastic film is used to cover window glass, which reduces the interior light intensity by 10%. By what percentage is the intensity reduced, if two layers are used?

4. Double-thickness colored window glass is supposed to reduce the interior light intensity by 20%. What is the reduction for single-thickness colored glass?

RC-Electric Circuits. In the exercises below, solve for $Q(t)$ when $Q_0 = 10$ and graph $Q(t)$ on $0 \le t \le 5$.

5. $R = 1$, $C = 0.01$.

6. $R = 0.05$, $C = 0.001$.

7. $R = 0.05$, $C = 0.01$.

8. $R = 5$, $C = 0.1$.

9. $R = 2$, $C = 0.01$.

10. $R = 4$, $C = 0.15$.

11. $R = 4$, $C = 0.02$.

12. $R = 50$, $C = 0.001$.

LR-Electric Circuits. In the exercises below, solve for $I(t)$ when $I_0 = 5$ and graph $I(t)$ on $0 \le t \le 5$.

13. $L = 1$, $R = 0.5$.

14. $L = 0.1$, $R = 0.5$.

15. $L = 0.1$, $R = 0.05$.

16. $L = 0.01$, $R = 0.05$.

17. $L = 0.2$, $R = 0.01$.

18. $L = 0.03$, $R = 0.01$.

19. $L = 0.05$, $R = 0.005$.

20. $L = 0.04$, $R = 0.005$.

Interest and Continuous Interest. Financial formulas which appear on page 17 are applied below, following the ideas in Examples 11, 12 and 13, pages 21–23.

21. **(Total Interest)** Compute the total daily interest and also the total continuous interest for a 10-year loan of $5,000$ dollars at 5% per annum.

22. **(Total Interest)** Compute the total daily interest and also the total continuous interest for a 15-year loan of $7,000$ dollars at $5\frac{1}{4}\%$ per annum.

23. **(Monthly Payment)** Find the monthly payment for a 3-year loan of $8,000$ dollars at 7% per annum compounded continuously.

24. **(Monthly Payment)** Find the monthly payment for a 4-year loan of $7,000$ dollars at $6\frac{1}{3}\%$ per annum compounded continuously.

25. **(Effective Yield)** Determine the effective annual yield for a certificate of deposit at $7\frac{1}{4}\%$ interest per annum, compounded continuously.

26. **(Effective Yield)** Determine the effective annual yield for a certificate of deposit at $5\frac{3}{4}\%$ interest per annum, compounded continuously.

27. **(Retirement Funds)** Assume a starting salary of $35,000$ dollars per year, which is expected to increase 3% per year. Retirement contributions are $10\frac{1}{2}\%$ of salary, deposited monthly, growing at $5\frac{1}{2}\%$ continuous interest per annum. Find the retirement amount after 30 years.

28. **(Retirement Funds)** Assume a starting salary of 45, 000 dollars per year, which is expected to increase 3% per year. Retirement contributions are $9\frac{1}{2}\%$ of salary, deposited monthly, growing at $6\frac{1}{4}\%$ continuous interest per annum. Find the retirement amount after 30 years.

29. **(Actual Cost)** A van is purchased for 18, 000 dollars with no money down. Monthly payments are spread over 8 years at $12\frac{1}{2}\%$ interest per annum, compounded continuously. What is the actual cost of the van?

30. **(Actual Cost)** Furniture is purchased for 15, 000 dollars with no money down. Monthly payments are spread over 5 years at $11\frac{1}{8}\%$ interest per annum, compounded continuously. What is the actual cost of the furniture?

Radioactive Decay. Assume the decay model $A' = -kA$ from page 18. Below, $A(T) = 0.5A(0)$ defines the *half-life* T. Methods parallel Examples 14– 17 on pages 24– 25.

31. **31.**(Half-Life) Determine the half-life of a radium sample which decays by 5.5% in 13 years.

32. **(Half-Life)** Determine the half-life of a radium sample which decays by 4.5% in 10 years.

33. **(Half-Life)** Assume a radioactive isotope has half-life 1800 years. Determine the percentage decayed after 150 years.

34. **(Half-Life)** Assume a radioactive isotope has half-life 1650 years. Determine the percentage decayed after 99 years.

35. **(Disintegration Constant)** Determine the constant k in the model $A' = -kA$ for radioactive material that disintegrates by 5.5% in 13 years.

36. **(Disintegration Constant)** Determine the constant k in the model $A' = -kA$ for radioactive material that disintegrates by 4.5% in 10 years.

37. **(Radiocarbon Dating)** A fossil found near the town of Dinosaur, Utah contains carbon-14 at a ratio of 6.21% to the atmospheric value. Determine its approximate age according to Libby's method.

38. **(Radiocarbon Dating)** A fossil found in Colorado contains carbon-14 at a ratio of 5.73% to the atmospheric value. Determine its approximate age according to Libby's method.

39. **(Radiocarbon Dating)** In 1950, the Lascaux Cave in France contained charcoal with 14.52% of the carbon-14 present in living wood samples nearby. Estimate by Libby's method the age of the charcoal sample.

40. **(Radiocarbon Dating)** At an excavation in 1960, charcoal from building material had 61% of the carbon-14 present in living wood nearby. Estimate the age of the building.

41. **(Percentage of an Isotope)** A radioactive isotope disintegrates by 5% in 12 years. By what percentage is it reduced in 99 years?

42. **(Percentage of an Isotope)** A radioactive isotope disintegrates by 6.5% in 1, 000 years. By what percentage is it reduced in 5, 000 years?

Chemical Reactions. Assume below the model $A' = kA$ for a first-order reaction. See page 19 and Example 18, page 25.

43. **(First-Order $A+B \longrightarrow C$)** A first order reaction produces product C from chemical A and catalyst B. Model the production of C, given $N\%$ of A remains after t_0 minutes.

44. **(First-Order $A+B \longrightarrow C$)** A first order reaction produces product C from chemical A and catalyst B. Model the production of C, given $M\%$ of A is depleted after t_0 minutes.

45. **(Law of Mass-Action)** Consider a second-order chemical reaction $X(t)$ with $k = 0.14$, $\alpha = 1$, $\beta = 1.75$, $X(0) = 0$. Find an explicit formula for $X(t)$ and graph it on $t = 0$ to $t = 2$.

46. **(Law of Mass-Action)** Consider a second-order chemical reaction $X(t)$ with $k = 0.015$, $\alpha = 1$, $\beta = 1.35$, $X(0) = 0$. Find an explicit formula for $X(t)$ and graph it on $t = 0$ to $t = 10$.

47. **(Derive Mass-Action Solution)** Let k, α, β be positive constants, $\alpha < \beta$. Solve $X' = k(\alpha - X)(\beta - X)$, $X(0) = X_0$ by the substitution $u = (\alpha - X)/(\beta - X)$, showing that $X = (\alpha - \beta u)/(1 - u)$, $u = u_0 e^{(\alpha-\beta)kt}$, $u_0 = (\alpha - X_0)/(\beta - X_0)$.

48. **(Verify Mass-Action Solution)** Let k, α, β be positive constants, $\alpha < \beta$. Define $X = (\alpha - \beta u)/(1 - u)$, where $u = u_0 e^{(\alpha-\beta)kt}$ and $u_0 = (\alpha - X_0)/(\beta - X_0)$. Verify by calculus computation that (1) $X' = k(\alpha - X)(\beta - X)$ and (2) $X(0) = X_0$.

Drug Dosage. Employ the drug dosage model $D(t) = D_0 e^{-ht}$ given on page 20. Let h be determined by a half-life of three hours. Apply the techniques of Example 19, page 26.

49. **(Injection Dosage)** Bloodstream injection of a drug into an animal requires a minimum of 20 milligrams per pound of body weight. Predict the dosage for a 12-pound animal which will maintain a drug level 3% higher than the minimum for two hours.

50. **(Injection Dosage)** Bloodstream injection of an antihistamine into an animal requires a minimum of 4 milligrams per pound of body weight. Predict the dosage for a 40-pound animal which will maintain an antihistamine level 5% higher than the minimum for twelve hours.

51. **(Oral Dosage)** An oral drug with first dose 250 milligrams is absorbed into the bloodstream after 45 minutes. Predict the number of hours after the first dose at which to take a second dose, in order to maintain a blood level of at least 180 milligrams for three hours.

52. **(Oral Dosage)** An oral drug with first dose 250 milligrams is absorbed into the bloodstream after 45 minutes. Determine three (small) dosage amounts, and their administration time, which keep the blood level above 180 milligrams but below 280 milligrams over three hours.