Name KEY Classtime 27 NOV 2006 Midterm Exam 3 – Ver 1: 7:30

1. **(ch4)** Complete enough of the following to add to 100%.

(a) [100%] Let V be the vector space of all continuous functions defined on $-2 \le x \le 2$. Define S to be the set of all functions $f(x)$ in V such that $f(0) = \int_{-2}^{2} f(|x|)dx$, $f(-1) = 0$. Prove that S is a subspace of V, by using the Subspace Criterion.

(b) [30%] Let V be the set of all 3×1 column vectors $\mathbf{x}$ with components x_1, x_2, x_3. Assume the usual $\mathcal{R}^3$ rules for addition and scalar multiplication. Let S be the subset of S defined by the equations $\mathbf{a} \cdot \mathbf{x} = \mathbf{b} \cdot \mathbf{x}$, where $\mathbf{a}$ and $\mathbf{b}$ are vectors in V. Prove that S is a subspace of V.

(c) [70%] Solve for the unknowns x_1, x_2, x_3, x_4 in the system of equations below by augmented matrix RREF methods, showing all details. Report the vector form of the general solution.

$$\begin{array}{rcrcrcrcl} x_1 & + & x_2 & - & 2x_3 & + & 3x_4 & = & 1 \\ & & x_2 & + & 2x_3 & & & = & 0 \\ x_1 & + & 2x_2 & & & + & 3x_4 & = & 1 \\ x_1 & + & 3x_2 & + & 2x_3 & + & 3x_4 & = & 1 \end{array}$$

(a) $\vec{0}$ is in S, because function $f(x)=0$ satisfies both equations. Let f_1, f_2 be in S. Then both equations are satisfied by f_1, f_2; if c_1, c_2 denote constants then $f = c_1 f_1 + c_2 f_2$ satisfies

I. $f(0) = c_1 f_1(0) + c_2 f_2(0)$
$= c_1 \int_{-2}^{2} f_1(|x|)dx + c_2 \int_{-2}^{2} f_2(|x|)dx$
$= \int_{-2}^{2} (c_1 f_1 + c_2 f_2)(|x|)dx$
$= \int_{-2}^{2} f(|x|)dx$

II. $f(-1) = c_1 f_1(-1) + c_2 f_2(-1)$
$= c_1(0) + c_2(0)$
$= 0$

By Thm 1, 4.2, Subspace Criterion, S is a subspace of V

(b) Let $\vec{c} = \vec{a} - \vec{b}$ so the restriction equation is $\vec{c} \cdot \vec{x} = \vec{0}$. Let matrix A have rows $\vec{c}, \vec{0}, \vec{0}$. Then $A\vec{x} = \vec{0}$ defines S. By Thm 2, 4.2 E&P, S is a subspace of V.

(c) $C = \left(\begin{array}{cccc|c} 1 & 1 & -2 & 3 & 1 \\ 0 & 1 & 2 & 0 & 0 \\ 1 & 2 & 0 & 3 & 1 \\ 1 & 3 & 2 & 3 & 1 \end{array}\right)$ is frame 1. Last frame = $\mathrm{rref}(C) = \left(\begin{array}{cccc|c} 1 & 0 & -4 & 3 & 1 \\ 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$

$\begin{cases} x_1 - 4x_3 + 3x_4 = 1 \\ x_2 + 2x_3 = 0 \\ 0 = 0 \\ 0 = 0 \end{cases}$ is the reduced echelon system. Free $= x_3, x_4$.

$\begin{cases} x_1 = 4t_1 - 3t_2 + 1 \\ x_2 = -2t_1 \\ x_3 = t_1 \\ x_4 = t_2 \end{cases}$ gen sol $\boxed{\vec{x} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + t_1 \begin{pmatrix} 4 \\ -2 \\ 1 \\ 0 \end{pmatrix} + t_2 \begin{pmatrix} -3 \\ 0 \\ 0 \\ 1 \end{pmatrix}}$

Use this page to start your solution. Staple extra pages as needed.

Name KEY Classtime 27 Nov 2006 Midterm Exam 3 – Ver 1: 7:30

2. **(ch5)** Complete (a), (b) and either (c) or (d). Do not do both (c) and (d).

(a) [30%] Given $3x''(t) + 8x'(t) + 2x(t) = 0$, which represents a damped spring-mass system with $m = 3$, $c = 8$, $k = 2$, **solve** the differential equation [20%] and **classify** the answer as over-damped, critically damped or under-damped [10%].

(b) [10%] Both undetermined coefficients and variation of parameters can solve $x'' + x' = t^2$. Without actually solving, which method is fastest? Explain your reasoning.

(c) [60%] Find by undetermined coefficients the steady-state periodic solution for the equation $x'' + 2x' + 5x = 5\sin(2t)$.

(d) [60%] If you did (c) above, then skip this one! Find by variation of parameters a particular solution x_p for the equation $x'' + 2x' + 5x = e^{t^2}\tan(t)$. To save time, don't try to evaluate integrals (it's impossible).

(a) $3r^2+8r+2=0$ has roots $r_1 = \frac{1}{3}(-4+\sqrt{10})$, $r_2 = \frac{1}{3}(-4-\sqrt{10})$. The gen sol is $x(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}$. over damped.

(b) undetermined coefficients is faster, because only polynomials appear. For var. of param., integrals like $\int t^2 e^{at}\,dt$ appear.

(c) trial $y = d_1 \cos 2t + d_2 \sin 2t$. Substitute. Then $d_1 = \frac{-20}{17}$, $d_2 = \frac{5}{17}$.

$y_p = \frac{-20}{17}\cos 2t + \frac{5}{17}\sin 2t$

(d) $x_1 = e^{-t}\cos 2t$, $x_2 = e^{-t}\sin 2t$ because the roots of $r^2+2r+5=0$ are $-1 \pm 2i$. Then $W = \begin{vmatrix} x_1 & x_2 \\ x_1' & x_2' \end{vmatrix} = 2e^{-2t}$. By (33) in E&P,

$$x_p(t) = u_1(t)x_1(t) + u_2(t)x_2(t)$$
$$= u_1 e^{-t}\cos 2t + u_2 e^{-t}\sin 2t$$
$$u_1 = -\int \frac{x_2 f}{W} = -\int \frac{e^{-t}\sin 2t\, e^{t^2}\tan(t)\,dt}{2e^{-2t}}$$
$$u_2 = \int \frac{x_1 f}{W} = \int \frac{e^{-t}\cos 2t\, e^{t^2}\tan(t)\,dt}{2e^{-2t}}$$

Use this page to start your solution. Staple extra pages as needed.

Name KEY Classtime 27 Nov 2006 **Midterm Exam 3 – Ver 1: 7:30**

3. **(ch5)** Complete all parts below.

(a) [75%] A non-homogeneous linear differential equation with constant coefficients has right side $f(x) = xe^{-x} + 3x^2 + x\sin x$ and characteristic equation $r(r+1)^2(r^2+1) = 0$. Determine the **corrected** trial solution for y_p according to the method of undetermined coefficients. To save time, **do not** evaluate the undetermined coefficients (that is, do undetermined coefficient steps $\boxed{1}$ and $\boxed{2}$, but skip steps $\boxed{3}$ and $\boxed{4}$)! Undocumented detail or guessing earns no credit.

(b) [25%] Using the *recipe* for higher order constant-coefficient differential equations, write out the general solution when the characteristic equation is $(r^3-4r)^2(r+2)(r^2+r+1) = 0$.

(a) trial $y = (d_1 + d_2 x + d_3 x^2) + (d_4 e^{-x} + d_5 x e^{-x})$
$+ \left((d_6 + d_7 x)\cos x + (d_8 + d_9 x)\sin x\right)$

There are 3 groups of related atoms.

$L = \{ e^{0x}, e^{-x}, xe^{-x}, \cos x, \sin x \}$ = atom list for the char eq.

corrected trial $y = (d_1 + d_2 x + d_3 x^2)x$
$+ (d_4 e^{-x} + d_5 x e^{-x})x^2$
$+ \left((d_6 + d_7 x)\cos x + (d_8 + d_9 x)\sin x\right)x$

(b) $r^2(r-2)^2(r+2)^2(r+2)(r^2+r+1) = 0$
$r^2(r-2)^2(r+2)^3(r^2+r+1) = 0$
roots $= 0, 0, 2, 2, -2, -2, -2, \frac{1}{2} \pm \frac{\sqrt{3}}{2}i$
$L = \{ 1, x, e^{2x}, xe^{2x}, e^{-2x}, xe^{-2x}, x^2e^{-2x}, e^{x/2}\cos\frac{\sqrt{3}x}{2}, e^{x/2}\sin\frac{\sqrt{3}x}{2} \}$

y = linear combination of the atoms in list L

$y = c_1 + c_2 x$
$+ c_3 e^{2x} + c_4 x e^{2x}$
$+ c_5 e^{-2x} + c_6 x e^{-2x} + c_7 x^2 e^{-2x}$
$+ c_8 e^{x/2}\cos(\sqrt{3}x/2) + c_9 e^{x/2}\sin(\sqrt{3}x/2)$

Use this page to start your solution. Staple extra pages as needed.

Name KEY Classtime 27 Nov 2006 **Midterm Exam 3 – Ver 1: 7:30**

4. **(ch6)** Complete all of the items below.

(a) [30%] Find the eigenvalues of the matrix $A = \begin{bmatrix} 5 & -2 & 1 & 4 \\ 2 & 5 & -3 & 5 \\ 0 & 0 & 0 & 7 \\ 0 & 0 & 0 & 5 \end{bmatrix}$. To save time, **do not** find eigenvectors!

(b) [70%] Given $A = \begin{bmatrix} 1 & 1 & -1 \\ 0 & 4 & 1 \\ 0 & 1 & 4 \end{bmatrix}$, then there exists an invertible matrix P and a diagonal matrix D such that $AP = PD$. Is $\mathbf{v} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ a possible column of P? Explain why or why not.

(a) Expand $A - \lambda I$ by cofactors along the last row. Repeat. Then

$$(5-\lambda)(0-\lambda)((5-\lambda)^2+4) = 0$$

$$\boxed{\lambda = 5, 0, 5 \pm 2i}$$

(b) It is a column $\vec{v}$ of $P \Leftrightarrow A\vec{v} = \lambda\vec{v}$ for some λ.

$$A\vec{v} = \begin{pmatrix} 1 & 1 & -1 \\ 0 & 4 & 1 \\ 0 & 1 & 4 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} 2 \\ 4 \\ 1 \end{pmatrix}$$

$$\neq \lambda \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \text{ for any } \lambda.$$

$\boxed{\text{No}}$

Use this page to start your solution. Staple extra pages as needed.

Name KEY Classtime 27 NOV 2006 Midterm Exam 3 – Ver 1: 7:30

5. **(ch6)** Complete all parts below.

Consider the 3×3 matrix

$$A = \begin{pmatrix} 3 & 1 & -1 \\ 0 & 3 & 1 \\ 0 & 1 & 3 \end{pmatrix}.$$

Already computed are eigenpairs

$$\left(3, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\right), \quad \left(4, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}\right).$$

(a) [40%] Compute and then display an invertible matrix P and a diagonal matrix D such that $AP = PD$.

(c) [30%] Describe precisely, and explicitly for A above, Fourier's model for the computation of $A\mathbf{x}$.

(c) [30%] Display the vector general solution $\mathbf{x}(t)$ of the linear differential system $\mathbf{x}' = A\mathbf{x}$.

(a) Eigenvalues satisfy $\det(A - \lambda I) = 0$. Then $(3-\lambda)((3-\lambda)^2 - 1) = 0$. So $\lambda = 3, 2, 4$. We are missing eigenpair $(2, \vec{v})$.

$B = A - 2I$
$= \begin{pmatrix} 1 & 1 & -1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}$
$\cong \begin{pmatrix} 1 & 1 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ combo
$\cong \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ combo

Last frame rref(B).

Free $= x_3$

$\begin{cases} x_1 = 2t_1 \\ x_2 = -t_1 \\ x_3 = t_1 \end{cases}$

$\vec{v} = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$

$P = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 0 & 1 & 1 \end{pmatrix} \quad D = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 2 \end{pmatrix}$ Then $AP = PD$, by Theorem.

(b) $A\left(c_1\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + c_2\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + c_3\begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}\right) = 3c_1\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + 4c_2\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + 2c_3\begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$

(c) $\vec{x}(t) = c_1 e^{3t}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + c_2 e^{4t}\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + c_3 e^{2t}\begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$

Use this page to start your solution. Staple extra pages as needed.