1. (ch4) Complete enough of the following to add to 100%.

(a) [100%] Let \( V \) be the vector space of all continuous functions defined on \(-2 \leq x \leq 2\). Define \( S \) to be the set of all functions \( f(x) \) in \( V \) such that \( f(0) = \int_{-2}^{2} f(|x|)dx \), \( f(-1) = 0 \). Prove that \( S \) is a subspace of \( V \), by using the Subspace Criterion.

(b) [30%] Let \( V \) be the set of all \( 3 \times 1 \) column vectors \( \mathbf{x} \) with components \( x_1, x_2, x_3 \). Assume the usual \( \mathbb{R}^3 \) rules for addition and scalar multiplication. Let \( S \) be the subset of \( S \) defined by the equations \( \mathbf{a} \cdot \mathbf{x} = \mathbf{b} \cdot \mathbf{x} \), where \( \mathbf{a} \) and \( \mathbf{b} \) are vectors in \( V \). Prove that \( S \) is a subspace of \( V \).

(c) [70%] Solve for the unknowns \( x_1, x_2, x_3, x_4 \) in the system of equations below by augmented matrix RREF methods, showing all details. Report the vector form of the general solution.

\[
\begin{align*}
x_1 &+ x_2 - 2x_3 + 3x_4 = 1 \\
x_2 &+ 2x_3 = 0 \\
x_1 &+ 2x_2 + 3x_4 = 1 \\
x_1 &+ 3x_2 + 2x_3 + 3x_4 = 1
\end{align*}
\]

\( \mathbf{b} \) is in \( S \), because function \( f(x) = 0 \) satisfies both equations. Let \( f_1, f_2 \) be in \( S \). Then both equations are satisfied by \( f_1, f_2 \); if \( c_1, c_2 \) denote constants then \( f = c_1 f_1 + c_2 f_2 \) satisfies:

I. \( f(0) = c_1 f_1(0) + c_2 f_2(0) \) 
   \[ = c_1 \int_{-2}^{0} f_1(1x)dx + c_2 \int_{-2}^{0} f_2(1x)dx \]
   \[ = \int_{-2}^{0} (c_1 f_1 + c_2 f_2)(1x)dx \]
   \[ = \int_{-2}^{0} f(1x)dx \]

II. \( f(-1) = c_1 f_1(-1) + c_2 f_2(-1) \) 
   \[ = c_1 (0) + c_2 (0) \]
   \[ = 0 \]

By Thm 1, 8.2, Subspace Criterion, \( S \) is a subspace of \( V \).

(b) Let \( \mathbf{c} = \overline{a} - \overline{b} \) so the restriction equation is \( \mathbf{c} \cdot \mathbf{x} = \overline{c} \). Let matrix \( \mathbf{A} \) have rows \( \mathbf{c}, \overline{b}, \overline{c} \). Then \( A\mathbf{x} = \overline{c} \) defines \( S \). By Thm 2, 8.2 EEP, \( S \) is a subspace of \( V \).

(c) \( \overline{c} = \begin{pmatrix} 1 & 1 & -2 & 3 \\ 0 & 1 & 2 & 0 \\ 1 & 3 & 2 & 3 \\ 1 & 1 & 1 & 1 \end{pmatrix} \) is frame 1. Last frame - \( \text{rref}(\overline{c}) = \begin{pmatrix} 1 & 0 & -4 & 3 & | & 1 \\ 0 & 1 & 2 & 0 & | & 0 \\ 0 & 0 & 0 & 0 & | & 0 \end{pmatrix} \)

\[
\begin{align*}
x_1 - 4x_3 + 3x_4 &= 0 \\
x_2 + 2x_3 &= 0 \\
x_1 - 4x_3 + 3x_4 &= 0 \\
x_2 &= -2t_1 \\
x_3 &= t_1 \\
x_4 &= t_2 \end{align*}
\]

is the reduced echelon system. Free = \( x_3, x_4 \).

Use this page to start your solution. Staple extra pages as needed.
2. (ch5) Complete (a), (b) and either (c) or (d). Do not do both (c) and (d).

(a) [30%] Given $3x''(t) + 8x'(t) + 2x(t) = 0$, which represents a damped spring-mass system with $m = 3$, $c = 8$, $k = 2$, **solve** the differential equation [20%] and **classify** the answer as over-damped, critically damped or under-damped [10%].

(b) [10%] Both undetermined coefficients and variation of parameters can solve $x'' + x' = t^2$. Without actually solving, which method is fastest? Explain your reasoning.

(c) [60%] Find by undetermined coefficients the steady-state periodic solution for the equation $x'' + 2x' + 5x = e^{t^2} \tan(t)$.

(d) [60%] If you did (c) above, then skip this one! Find by variation of parameters a particular solution $x_p$ for the equation $x'' + 2x' + 5x = e^{t^2} \tan(t)$. To save time, don’t try to evaluate integrals (it’s impossible).

(a) $3r^2 + 8r + 2 = 0$ has roots $r_1 = \frac{1}{3}(-4 + \sqrt{10})$, $r_2 = \frac{1}{3}(-4 - \sqrt{10})$. The general solution is $x(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}$. Over-damped.

(b) Undetermined coefficients is faster, because only polynomials appear. For vary ofparam., integrals like $\int t^2 e^{at} dt$ appear:

(c) trial $y = d_1 \cos 2t + d_2 \sin 2t$. Substitute. Then $d_1 = -\frac{25}{17}$, $d_2 = \frac{5}{17}$.

\[ y_p = -\frac{25}{17} \cos 2t + \frac{5}{17} \sin 2t \]

(d) $x_1 = e^t \cos 2t$, $x_2 = e^t \sin 2t$ because the roots of $r^2 + 2r + 5 = 0$ are $-1 \pm 2i$. Then $W = \left| \begin{array}{cc} x_1 & x_2 \\ x_1' & x_2' \end{array} \right| = 2e^{2t}$. By (33) in $E \in \mathbb{R}$,

\[
x_p(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = u_1(t) x_1(t) + u_2(t) x_2(t) = u_1 e^t \cos 2t + u_2 e^t \sin 2t
\]

\[
u_1 = -\int \frac{x_2 f}{W} = -\int \frac{e^t \sin 2t e^t \tan(t) dt}{2e^{-2t}}
\]

\[
u_2 = \int \frac{x_1 f}{W} = \int \frac{e^t \cos 2t e^t \tan(t) dt}{2e^{-2t}}
\]

Use this page to start your solution. Staple extra pages as needed.
3. (ch5) Complete all parts below.

(a) [75%] A non-homogeneous linear differential equation with constant coefficients has right side \( f(x) = xe^{-x} + 3x^2 + x \sin x \) and characteristic equation \( r(r+1)^3(r^2+1) = 0 \). Determine the **corrected** trial solution for \( y_p \) according to the method of undetermined coefficients. To save time, **do not** evaluate the undetermined coefficients (that is, do undetermined coefficient steps [1] and [2], but skip steps [3] and [4])! Undocumented detail or guessing earns no credit.

(b) [25%] Using the recipe for higher order constant-coefficient differential equations, write out the general solution when the characteristic equation is \( (r^3 - 4r^2)(r+2)(r^2 + r + 1) = 0 \).

\[
\begin{align*}
\text{Corrected trial:} & \quad y = (d_1 + d_2 x + d_3 x^2) e^{-x} + (d_4 e^{-x} + d_5 x e^{-x}) x + (d_6 + d_7 x) \cos x + (d_8 + d_9 x) \sin x \\
\text{or} & \quad (d_1 + d_2 x + d_3 x^2) e^{-x} + (d_4 e^{-x} + d_5 x e^{-x}) x + (d_6 + d_7 x) \cos x + (d_8 + d_9 x) \sin x \\
\end{align*}
\]

Use this page to start your solution. Staple extra pages as needed.
4. (ch6) Complete all of the items below.

(a) [30%] Find the eigenvalues of the matrix \( A = \begin{bmatrix} 5 & -2 & 1 & 4 \\ 2 & 5 & -3 & 5 \\ 0 & 0 & 0 & 7 \\ 0 & 0 & 0 & 5 \end{bmatrix} \). To save time, do not find eigenvectors!

(b) [70%] Given \( A = \begin{bmatrix} 1 & 1 & -1 \\ 0 & 4 & 1 \\ 0 & 1 & 4 \end{bmatrix} \), then there exists an invertible matrix \( P \) and a diagonal matrix \( D \) such that \( AP = PD \). Is \( \mathbf{v} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \) a possible column of \( P \)? Explain why or why not.

(a) Expand \( A - \lambda I \) by cofactors along the last row. Repeat. Then

\[
\lambda = 5, 0, 5 \pm 2i
\]

(b) It is a column \( \mathbf{v} \) of \( P \) \( \iff \) \( A \mathbf{v} = \lambda \mathbf{v} \) for some \( \lambda \).

\[
A \mathbf{v} = \begin{bmatrix} 1 & 1 & -1 \\ 0 & 4 & 1 \\ 0 & 1 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ 1 \end{bmatrix} = \lambda \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}
\]

\[\text{No}\]
5. (ch6) Complete all parts below.
Consider the $3 \times 3$ matrix

\[
A = \begin{pmatrix}
3 & 1 & -1 \\
0 & 3 & 1 \\
0 & 1 & 3 \\
\end{pmatrix}.
\]

Already computed are eigenpairs

\[
\begin{pmatrix} 3, \\
1 \\
0 \\
\end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 4, \\
0 \\
1 \\
\end{pmatrix}.
\]

(a) [40%] Compute and then display an invertible matrix $P$ and a diagonal matrix $D$ such that $AP = PD$.

(b) [30%] Describe precisely, and explicitly for $A$ above, Fourier’s model for the computation of $Ax$.

(c) [30%] Display the vector general solution $x(t)$ of the linear differential system $x' = Ax$.

\[\mathbf{B} = A - 2I = \begin{pmatrix}
1 & 1 & -1 \\
0 & 1 & 1 \\
0 & 1 & 1 \\
\end{pmatrix} \text{ combo } \begin{pmatrix} 0 \\
-2 \\
1 \end{pmatrix} \text{ last form ref } (B).
\]

\[\mathbf{P} = \begin{pmatrix}
1 & 0 & 2 \\
0 & 1 & -1 \\
0 & 0 & 1 \\
\end{pmatrix} \text{ and } PAP = PDP, \text{ by Theorem.}
\]

\[\mathbf{A} \begin{pmatrix} c_1 \\
0 \\
0 \end{pmatrix} + \mathbf{A} \begin{pmatrix} 0 \\
1 \\
0 \end{pmatrix} + \mathbf{A} \begin{pmatrix} 0 \\
0 \\
2 \end{pmatrix} = \begin{pmatrix} 3c_1 \\
0 \\
1 \end{pmatrix} + \begin{pmatrix} 4c_2 \\
0 \\
2 \end{pmatrix} + \begin{pmatrix} 2c_3 \\
0 \\
1 \end{pmatrix}
\]

\[\mathbf{x}(t) = c_1 e^{3t} \begin{pmatrix} 1 \\
0 \\
0 \end{pmatrix} + c_2 e^{4t} \begin{pmatrix} 0 \\
1 \\
0 \end{pmatrix} + c_3 e^{2t} \begin{pmatrix} 2 \\
0 \\
1 \end{pmatrix}
\]

Use this page to start your solution. Staple extra pages as needed.