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1. (ch4) Complete enough of the following to add to 100%.

(a) [100%] Let V be the vector space of all continuous functions defined on $-\infty < x < \infty$. Define S to be the set of all functions $f(x)$ in V such that $f(0) = \int_0^1 f(2x)dx$, $f(1) = f(2)$. Prove that S is a subspace of V, by using the Subspace Criterion.

(b) [30%] Let V be the set of all 3×1 column vectors $\mathbf{x}$ with components x_1, x_2, x_3. Assume the usual $\mathcal{R}^3$ rules for addition and scalar multiplication. Let S be the subset of V defined by the equations $\mathbf{a} \cdot \mathbf{x} = 0$, $\mathbf{b} \cdot \mathbf{x} = 0$, $x_1 - x_2 + 2x_3 = 0$ where $\mathbf{a}$ and $\mathbf{b}$ are vectors in V. Prove that S is a subspace of V.

(c) [70%] Solve for the unknowns x_1, x_2, x_3, x_4 in the system of equations below by augmented matrix RREF methods, showing all details. Report the vector form of the general solution.

$$\begin{array}{rcrcrcrcl} x_1 & + & x_2 & - & 2x_3 & + & 3x_4 & = & 1 \\ & & 2x_2 & + & 3x_3 & & & = & 1 \\ x_1 & + & 3x_2 & + & x_3 & + & 3x_4 & = & 2 \\ 3x_1 & + & 7x_2 & + & & & 9x_4 & = & 5 \end{array}$$

(a) Vector $\vec{0}$ is defined by equation $f(x)=0$. Both equations hold for f, then $\vec{0}$ is in S.
Given f_1, f_2 in S and constants c_1, c_2, then $f = c_1 f_1 + c_2 f_2$ satisfies

$$\begin{aligned} \text{I.}\quad f(0) &= c_1 f_1(0) + c_2 f_2(0) \\ &= c_1 \int_0^1 f_1(2x)dx + c_2 \int_0^1 f_2(2x)dx \\ &= \int_0^1 (c_1 f_1 + c_2 f_2)(2x)dx \\ &= \int_0^1 f(2x)dx, \\ \text{II.}\quad f(1) &= c_1 f_1(1) + c_2 f_2(1) \\ &= c_1 f_1(2) + c_2 f_2(2) \\ &= f(2). \end{aligned}$$

By the subspace criterion, Thm 2 in 4.2 E&P, S is a subspace of V.

(b) Let $\vec{c} = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$. Let A be 3×3 with rows $\vec{a}, \vec{b}, \vec{c}$. The 3 relations are equivalent to $A\vec{x} = \vec{0}$. By Thm 2, 4.2 in E&P, S is a subspace of V.

(c) $$C = \left(\begin{array}{cccc|c} 1 & 1 & -2 & 3 & 1 \\ 0 & 2 & 3 & 0 & 1 \\ 1 & 3 & 1 & 3 & 2 \\ 3 & 7 & 0 & 9 & 5 \end{array}\right) \qquad \mathrm{rref}(C) = \left(\begin{array}{cccc|c} 1 & 0 & -7/2 & 3 & 1/2 \\ 0 & 1 & 3/2 & 0 & 1/2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

= Frame 1 of sequence = Last frame of seq.

$$\left\{\begin{array}{l} x_1 - 7x_3/2 + 3x_4 = 1/2 \\ x_2 + 3x_3/2 = 1/2 \\ 0 = 0 \\ 0 = 0 \end{array}\right. \qquad \left\{\begin{array}{l} x_1 = 7t_1/2 - 3t_2 + 1/2 \\ x_2 = -3t_1/2 + 1/2 \\ x_3 = t_1 \\ x_4 = t_2 \end{array}\right. \qquad \vec{x} = \begin{pmatrix} 1/2 \\ 1/2 \\ 0 \\ 0 \end{pmatrix} + t_1 \begin{pmatrix} 7/2 \\ -3/2 \\ 1 \\ 0 \end{pmatrix} + t_2 \begin{pmatrix} -3 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

Reduced echelon system gen. sol.

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2. **(ch5)** Complete (a), (b), (c) and either (d) or (e). Do not do both (d) and (e).

(a) [30%] Given $6x''(t) + 31x'(t) + 35x(t) = 0$, which represents a damped spring-mass system with $m = 6$, $c = 31$, $k = 35$. **Solve** the differential equation [20%]. **Classify** the answer as over-damped, critically damped or under-damped [10%].

(b) [10%] Undetermined coefficients can solve $x'' + 2x' = e^{-2t}$. Without actually solving it, is it true or false that $x = d_1 e^{-2t}$ is the corrected trial solution? Explain your reasoning.

(c) [10%] Undetermined coefficients cannot solve $x'' + 2x' = \frac{t}{1+t^2}$. Why not?

(d) [50%] Find by undetermined coefficients the steady-state periodic solution for the equation $x''+2x' = 2\cos(4t)$.

(e) [50%] If you did (d) above, then skip this one! Display using variation of parameters a particular solution x_p for the equation $x'' + 2x' = e^{t^2}\ln|t|$. To save time, don't try to evaluate integrals (it's impossible).

(a) $6r^2+31r+35=0$
$(3r+5)(2r+7)=0$
$r=-5/3, -7/2$

over damped
$x(t) = c_1 e^{-5t/3} + c_2 e^{-7t/2}$

(b) NO. False. Coefficient d_1 vanishes upon substitution, because e^{-2t} is an atom of the homogeneous DE $x''+2x'$ (roots $-2, 0$).

(c) Undetermined coefficients works only when the DE has constant coefficients and $f(t)$ = linear combination of atoms: $f(t) = \frac{t}{1+t^2}$ is not such.

(d) $x = d_1 \cos 4t + d_2 \sin 4t$ = trial sol, no fixup rule
$d_1 = -1/10$, $d_2 = 1/20$
$x_h = c_1 + c_2 e^{-2t}$ is not part of the periodic sol if $c_2 \neq 0$, because e^{-2t} has limit zero at ∞ (not periodic). However, any c_1 can be added. Then

$$x(t) = c_1 + \left(-\tfrac{1}{10}\right)\cos 4t + \left(\tfrac{1}{20}\right)\sin 4t$$

(e) roots of $r^2+2r=0$ are $r=0,-2$, $L=\{1, e^{-2t}\}$ = atom list,
$x_1 = 1$, $x_2 = e^{-2t}$, $W = -2e^{-2t}$, $f(t) = e^{t^2}\ln|t|$. Then

$$x_p(t) = x_1 u_1 + x_2 u_2$$
$$u_1 = -\int \frac{x_2 f}{W} = -\int \frac{e^{-2t} f}{-2e^{-2t}} = \frac{1}{2}\int e^{t^2}\ln|t|\,dt$$
$$u_2 = \int \frac{x_1 f}{W} = \int \frac{f}{-2e^{-2t}} = \frac{-1}{2}\int e^{t^2+2t}\ln|t|\,dt$$

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3. **(ch5)** Complete all parts below.

(a) [75%] A non-homogeneous linear differential equation with constant coefficients has right side $f(x) = xe^x + e^{-x} + x^3 - 2x + 4 + e^{-x}\cos x$ and characteristic equation $r^3(r-1)^2(r^2+2r+2) = 0$. Determine the **corrected** trial solution for y_p according to the method of undetermined coefficients. To save time, **do not** evaluate the undetermined coefficients (that is, do undetermined coefficient steps $\boxed{1}$ and $\boxed{2}$, but skip steps $\boxed{3}$ and $\boxed{4}$)! Undocumented detail or guessing earns no credit.

(b) [25%] Using the *recipe* for higher order constant-coefficient differential equations, write out the general solution when the characteristic equation is $(r^3+4r^2)^2(r^2-16)(r^2+r)(r^2+1) = 0$.

(a) char eq roots $= 0, 0, 0, 1, 1, -1 \pm i$

$L = \{1, x, x^2, e^x, xe^x, e^{-x}\cos x, e^{-x}\sin x\}$

$$y = (d_1 + d_2 x + d_3 x^2 + d_4 x^3)x^3$$
$$+ (d_5 e^x + d_6 x e^x)x^2$$
$$+ d_7 e^{-x} \qquad \text{No fixup}$$
$$+ (d_8 e^{-x}\cos x + d_9 e^{-x}\sin x)x$$

(b) $r^4(r+4)^2(r-4)(r+4)\,r(r+1)(r^2+1) = 0$

$r^5(r+4)^3(r-4)(r+1)(r^2+1)$

$L = \{1, x, x^2, x^3, x^4, e^{-4x}, xe^{-4x}, x^2e^{-4x}, e^{4x}, e^{-x}, \cos x, \sin x\}$

$y =$ linear combination of the atoms in List L

$$y = c_1 + c_2 x + c_3 x^2 + c_4 x^3 + c_5 x^4$$
$$+ c_6 e^{-4x} + c_7 xe^{-4x} + c_8 x^2 e^{-4x}$$
$$+ c_9 e^{4x}$$
$$+ c_{10} e^{-x}$$
$$+ c_{11}\cos x + c_{12}\sin x$$

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4. **(ch6)** Complete all of the items below.

(a) [30%] Find the eigenvalues of the matrix $A = \begin{bmatrix} 4 & -3 & 1 & 4 \\ 3 & 4 & -3 & 5 \\ 0 & 0 & 0 & 9 \\ 0 & 0 & 9 & 0 \end{bmatrix}$. To save time, **do not** find eigenvectors!

(b) [70%] Given a certain 3×3 matrix A, the equation $AP = PD$ holds, where

$$P = \begin{pmatrix} -1 & 0 & 1 \\ -1 & 1 & 0 \\ 1 & 1 & 0 \end{pmatrix}, \quad D = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}.$$

Circle all possible eigenpairs of A.

$$\left(2, \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}\right), \quad \left(2, \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix}\right),$$

$$\left(2, \begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix}\right), \quad \left(3, \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}\right).$$

(a) Expand by cofactors along the bottom row of $|A - \lambda I|$ to get

$$|A - \lambda I| = -9 \begin{vmatrix} 4-\lambda & -3 & 4 \\ 3 & 4-\lambda & 5 \\ 0 & 0 & 9 \end{vmatrix} + (-\lambda) \begin{vmatrix} 4-\lambda & -3 & 1 \\ 3 & 4-\lambda & -3 \\ 0 & 0 & -\lambda \end{vmatrix}$$

$$= -81((4-\lambda)^2 + 9) + (-\lambda)(-\lambda)((4-\lambda)^2 + 9)$$

$$= (\lambda^2 - 81)((4-\lambda)^2 + 9)$$

Eigenvalues $= 9, -9, \ 4 \pm 3i$

(b) The pair for $\lambda = 3$ can't be an eigenpair, because $\begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix} \neq c \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$.
The other three can be tested at once by seeing if each is a linear combination of the two given eigenvectors, col(P,1) and col(P,2).
We find the pivot columns of

$$C = \begin{pmatrix} -1 & 0 & 1 & -1 & 1 \\ -1 & 1 & 2 & 0 & 3 \\ 1 & 1 & 0 & 2 & 1 \end{pmatrix}$$

are cols $= 1, 2$. The non-pivot columns of C are linear combinations of cols 1, 2. So all pairs for $\lambda = 2$ are eigenpairs.

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5. **(ch6)** Complete all parts below.

Consider the 3×3 matrix

$$A = \begin{pmatrix} 4 & 1 & -1 \\ 0 & 4 & 1 \\ 0 & 1 & 4 \end{pmatrix}.$$

Already computed are eigenpairs

$$\left(4, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\right), \quad \left(3, \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}\right).$$

(a) [25%] Compute and then display an invertible matrix P and a diagonal matrix D such that $AP = PD$.

(b) [25%] Describe precisely, and explicitly for A above, Fourier's model for the computation of $A\mathbf{x}$.

(c) [25%] Display the vector general solution $\mathbf{x}(t)$ of the linear differential system $\mathbf{x}' = A\mathbf{x}$.

(d) [25%] Give an example of an upper triangular 3×3 matrix A for which Fourier's model does not hold.

(a) $|A - \lambda I| = (4-\lambda)((4-\lambda)^2 - 1) = (4-\lambda)(3-\lambda)(5-\lambda)$ roots $= 3, 4, 5$

we are missing $\lambda = 5$. Solve $(A - 5I)\vec{v} = \vec{0}$ to get

$$\vec{v} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}.$$

Then $P = \begin{pmatrix} 1 & 2 & 0 \\ 0 & -1 & 1 \\ 0 & 1 & 1 \end{pmatrix}$ and $D = \begin{pmatrix} 4 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{pmatrix}$

(b) $A\left(c_1\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + c_2\begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} + c_3\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}\right) = 4c_1\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + 3c_2\begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} + 5c_3\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$

(c) $\vec{x}(t) = c_1 e^{4t}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + c_2 e^{3t}\begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} + c_3 e^{5t}\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$

(d) $A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$ has eigenvalues $1, 1, 1$; $A - 1 \cdot I = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$

has only one free variable, hence exactly one eigenpair.

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