Name KEY Classtime 28 Nov 2006 **Midterm Exam 3 – Ver 2: 10:45**

1. (ch4) Complete enough of the following to add to 100%.

(a) [100%] Let V be the vector space of all continuous functions defined on $-1 \le x \le 1$. Define S to be the set of all functions $f(x)$ in V such that $f(0) = \int_{-1}^{1} f(0.5 + |x/2|)dx$, $f(0.5) = 0$. Prove that S is a subspace of V, by using the Subspace Criterion.

(b) [30%] Let V be the set of all 3×1 column vectors $\mathbf{x}$ with components x_1, x_2, x_3. Assume the usual $\mathcal{R}^3$ rules for addition and scalar multiplication. Let S be the subset of S defined by the equations $2\mathbf{a} \cdot \mathbf{x} = \mathbf{b} \cdot \mathbf{x}$, where $\mathbf{a}$ and $\mathbf{b}$ are vectors in V. Prove that S is a subspace of V.

(c) [70%] Solve for the unknowns x_1, x_2, x_3, x_4 in the system of equations below by augmented matrix RREF methods, showing all details. Report the vector form of the general solution.

$$\begin{array}{rcrcrcrcl} x_1 & + & x_2 & - & 2x_3 & + & 3x_4 & = & 1 \\ & & x_2 & + & 2x_3 & & & = & 1 \\ x_1 & + & 2x_2 & + & & & 3x_4 & = & 2 \\ x_1 & + & 3x_2 & + & 2x_3 & + & 3x_4 & = & 3 \end{array}$$

(a) Define $g(x) = 0.5 + |x/2|$. Vector $\vec{0}$ is in S, because $f(x) = 0$ satisfies both equations. Let f_1, f_2 be in S and c_1, c_2 constants. Define $f = c_1 f_1 + c_2 f_2$. Then

I. $f(0) = c_1 f_1(0) + c_2 f_2(0)$

$= c_1 \int_{-1}^{1} f_1(g) + c_2 \int_{-1}^{1} f_2(g)$

$= \int_{-1}^{1} c_1 f_1(g) + c_2 f_2(g)$

$= \int_{-1}^{1} f(g)$

II. $f(0.5) = c_1 f_1(0.5) + c_2 f_2(0.5)$

$= c_1(0) + c_2(0)$

$= 0$

By Theorem 1, 4.2 E&P, Subspace Criterion, S is a subspace of V.

(b) Let $\vec{c} = 2\vec{a} - \vec{b}$. Then $\vec{c} \cdot \vec{x} = \vec{0}$ defines S. Let A be the matrix whose rows are $\vec{c}, \vec{0}, \vec{0}$. Then $A\vec{x} = \vec{0}$ defines S. By Theorem 2, 4.2 E&P, S is a subspace of V.

(c) $C = \left(\begin{array}{cccc|c} 1 & 1 & -2 & 3 & 1 \\ 0 & 1 & 2 & 0 & 1 \\ 1 & 2 & 0 & 3 & 2 \\ 1 & 3 & 2 & 3 & 3 \end{array}\right)$ Frame 1. Last Frame $\mathrm{rref}(C) = \left(\begin{array}{cccc|c} 1 & 0 & -4 & 3 & 0 \\ 0 & 1 & 2 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$

Reduced Echelon System $\left\{\begin{array}{rl} x_1 - 4x_3 + 3x_4 &= 0 \\ x_2 + 2x_3 &= 1 \\ 0 &= 0 \\ 0 &= 0 \end{array}\right.$ Free $= x_3, x_4$

Gen Sol $\left\{\begin{array}{l} x_1 = 4t_1 - 3t_2 \\ x_2 = 1 - 2t_1 \\ x_3 = t_1 \\ x_4 = t_2 \end{array}\right.$

$$\vec{x} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} + t_1 \begin{pmatrix} 4 \\ -2 \\ 1 \\ 0 \end{pmatrix} + t_2 \begin{pmatrix} -3 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

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2. **(ch5)** Complete (a), (b) and either (c) or (d). Do not do both (c) and (d).

(a) [30%] Given $x''(t) + 2x'(t) + 3x(t) = 0$, which represents a damped spring-mass system with $m = 1$, $c = 2$, $k = 3$, **solve** the differential equation [20%] and **classify** the answer as over-damped, critically damped or under-damped [10%].

(b) [10%] Both undetermined coefficients and variation of parameters can solve $x'' + x' = e^{-t}$. Without actually solving, is one method faster? Explain your reasoning.

(c) [60%] Find by undetermined coefficients the steady-state periodic solution for the equation $x'' + 2x' + 5x = 5\sin(3t)$.

(d) [60%] If you did (c) above, then skip this one! Find by variation of parameters a particular solution x_p for the equation $x'' + 2x' + 5x = e^{t^2}\cot(t^2)$. To save time, don't try to evaluate integrals (it's impossible).

(a) $r^2+2r+3=0 \quad r = -1 \pm \sqrt{2}\,i$ under damped

$x(t) = c_1 e^{-t}\cos(\sqrt{2}t) + c_2 e^{-t}\sin(\sqrt{2}t)$

(b) Variation of parameters involves integration of exponentials. It is easy. Both are fast, but a fixup rule slows down undetermined coefficients. I would do var of parameters first.

(c) trial $y = d_1\cos 3t + d_2 \sin 3t$. No fixup. Stuff into DE, solve for $d_1 = \frac{-15}{26}$, $d_2 = \frac{-5}{13}$. Then $x_p(t)$ = steady-state periodic sol

$= \left(\frac{-15}{26}\right)\cos 3t + \left(\frac{-5}{13}\right)\sin 3t$

(d) $r^2+2r+5=0$, roots $-1\pm 2i$. $x_1 = e^{-t}\cos 2t$, $x_2 = e^{-t}\sin 2t$,

$W = \begin{vmatrix} x_1 & x_2 \\ x_1' & x_2' \end{vmatrix} = 2e^{-2t}$. Then

$$\begin{aligned} x_p(t) &= u_1(t)\,x_1(t) + u_2(t)\,x_2(t) \\ &= u_1 e^{-t}\cos 2t + u_2 e^{-t}\sin 2t \end{aligned}$$

$$u_1 = -\int \frac{x_2 f}{W} = -\int \frac{e^{-t}\sin 2t\, e^{t^2}\cos(t^2)\,dt}{2e^{-2t}}$$

$$u_2 = \int \frac{x_1 f}{W} = \int \frac{e^{-t}\cos 2t\, e^{t^2}\cos(t^2)\,dt}{2e^{-2t}}$$

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3. **(ch5)** Complete all parts below.

(a) [75%] A non-homogeneous linear differential equation with constant coefficients has right side $f(x) = xe^{-x} + 2x^3 - x + 5 + x\cos x$ and characteristic equation $r^2(r+1)^3(r^2+4) = 0$. Determine the **corrected** trial solution for y_p according to the method of undetermined coefficients. To save time, **do not** evaluate the undetermined coefficients (that is, do undetermined coefficient steps $\boxed{1}$ and $\boxed{2}$, but skip steps $\boxed{3}$ and $\boxed{4}$)! Undocumented detail or guessing earns no credit.

(b) [25%] Using the *recipe* for higher order constant-coefficient differential equations, write out the general solution when the characteristic equation is $(r^3+4r)^2(r^2+2r)(r^2+r) = 0$.

(a) atoms of f : $1, x, x^2, x^3, e^{-x}, xe^{-x}, \cos x, \sin x, x\cos x, x\sin x$

trial y = linear combination of these atoms

Corrected trial $y = (d_1 + d_2 x + d_3 x^2 + d_4 x^3)x^2$ ← in L; $x^s = x^2$

$+ (d_5 e^{-x} + d_6 xe^{-x})x^3$ ← in L; $x^s = x^3$

$+ ((d_7 + d_8 x)\cos x + (d_9 + d_{10} x)\sin x)$ ← not in L

char eq roots $= 0, 0, -1, -1, -1, 2i, -2i$

$L = \{1, x, e^{-x}, xe^{-x}, x^2e^{-x}, \cos 2x, \sin 2x\}$

(b) $r^2(r^2+4)^2 r(r+2)r(r+1) = 0$

$r^4(r+1)(r+2)(r^2+4)^2$

roots $= 0, 0, 0, 0, -1, -2, 2i, 2i, -2i, -2i$

$L = \{1, x, x^2, x^3, e^{-x}, e^{-2x}, \cos 2x, \sin 2x, x\cos 2x, x\sin 2x\}$

y = linear combination of atoms in L

$= c_1 + c_2 x + c_3 x^2 + c_4 x^3$

$+ c_5 e^{-x}$

$+ c_6 e^{-2x}$

$+ (c_7 + c_8 x)\cos 2x + (c_9 + c_{10} x)\sin 2x$

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4. **(ch6)** Complete all of the items below.

(a) [30%] Find the eigenvalues of the matrix $A = \begin{bmatrix} 4 & -3 & 1 & 4 \\ 3 & 4 & -3 & 5 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 9 \end{bmatrix}$. To save time, **do not** find eigenvectors!

(b) [70%] Given $A = \begin{bmatrix} 1 & 1 & -1 \\ 0 & 4 & 1 \\ 0 & 1 & 4 \end{bmatrix}$, then there exists an invertible matrix P and a diagonal matrix D such that $AP = PD$. Which of the following is a possible column of P?

$$\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ -1 \end{pmatrix}.$$

(a) Expand $\det(A - \lambda I)$ along row 4. Then repeat.

$$(9-\lambda)(0-\lambda)((4-\lambda)^2+9) = 0$$

$$\boxed{\text{Roots} = 9, 0, 4 \pm 3i}$$

(b) a Column $\vec{v}$ of P satisfies $A\vec{v} = \lambda\vec{v}$ for some λ.

$$A\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \\ 1 \end{pmatrix} \neq \lambda \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

$$A\begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 5 \\ 5 \end{pmatrix} \neq \lambda \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$$

$$A\begin{pmatrix} 0 \\ -1 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ -5 \\ -5 \end{pmatrix} = (5)\begin{pmatrix} 0 \\ -1 \\ -1 \end{pmatrix}$$

It works. $\boxed{\begin{pmatrix} 0 \\ -1 \\ -1 \end{pmatrix} \text{ could be a Col of } P}$

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5. **(ch6)** Complete all parts below.

Consider the 3×3 matrix

$$A = \begin{pmatrix} 4 & 1 & -1 \\ 0 & 3 & 1 \\ 0 & 1 & 3 \end{pmatrix}.$$

Already computed are eigenpairs

$$\left(2, \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}\right), \quad \left(4, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\right).$$

(a) [40%] Compute and then display an invertible matrix P and a diagonal matrix D such that $AP = PD$.

(c) [30%] Describe precisely, and explicitly for A above, Fourier's model for the computation of $A\mathbf{x}$.

(c) [30%] Display the vector general solution $\mathbf{x}(t)$ of the linear differential system $\mathbf{x}' = A\mathbf{x}$.

(a) Eigenvalues satisfy $\det(A-\lambda I)=0$. This is $(4-\lambda)((3-\lambda)^2-1)=0$. roots = 4, 2, 4. We are missing eigenpair $(4, \vec{v})$.

$B = A - 4I = \begin{pmatrix} 0 & 1 & -1 \\ 0 & -1 & 1 \\ 0 & 1 & -1 \end{pmatrix} \cong \begin{pmatrix} 0 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ combo combo

$\left\{ \begin{array}{l} x_2 - x_3 = 0 \\ 0 = 0 \\ 0 = 0 \end{array} \right.$ x_1, x_3 are free. Reduced echelon Sys

$\left\{ \begin{array}{l} x_1 = t_1 \\ x_2 = t_2 \\ x_3 = t_2 \end{array} \right.$ Gen Sol $\vec{x}$

$\frac{\partial \vec{x}}{\partial t_1} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $\frac{\partial \vec{x}}{\partial t_2} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = \vec{v}$!

Third eigenpair $= \left(4, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}\right)$. $P = \begin{pmatrix} 1 & 1 & 0 \\ -1 & 0 & 1 \\ 1 & 0 & 1 \end{pmatrix}$ $D = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{pmatrix}$. Then $AP = PD$.

(b) $A\left(c_1 \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + c_3 \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}\right) = 2c_1 \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} + 4c_2 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + 4c_3 \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$

(c) $\vec{x}(t) = c_1 e^{2t} \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} + c_2 e^{4t} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + c_3 e^{4t} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$

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