1. (ch4) Complete enough of the following to add to 100%.

(a) [100%] Let \( V \) be the vector space of all continuous functions defined on \(-1 \leq x \leq 1\). Define \( S \) to be the set of all functions \( f(x) \) in \( V \) such that \( f(0) = \int_{-1}^{1} f(0.5 + |x/2|)dx, f(0.5) = 0 \). Prove that \( S \) is a subspace of \( V \), by using the Subspace Criterion.

(b) [30%] Let \( V \) be the set of all \( 3 \times 1 \) column vectors \( \mathbf{x} \) with components \( x_1, x_2, x_3 \). Assume the usual \( \mathbb{R}^3 \) rules for addition and scalar multiplication. Let \( S \) be the subset of \( S \) defined by the equations \( 2a \cdot \mathbf{x} = b \cdot \mathbf{x} \), where \( a \) and \( b \) are vectors in \( V \). Prove that \( S \) is a subspace of \( V \).

(c) [70%] Solve for the unknowns \( x_1, x_2, x_3, x_4 \) in the system of equations below by augmented matrix RREF methods, showing all details. Report the vector form of the general solution.

\[
\begin{align*}
x_1 + x_2 - 2x_3 + 3x_4 &= 1 \\
x_2 + 2x_3 &= 1 \\
x_1 + 2x_2 + 3x_4 &= 2 \\
x_1 + 3x_2 + 2x_3 + 3x_4 &= 3
\end{align*}
\]

Define \( g(x) = 0.5 + 1x/2 \). Vector \( \mathbf{0} \) is in \( S \), because \( f(x) = 0 \) satisfies both equations. Let \( f_1, f_2 \) be in \( S \) and \( c_1, c_2 \) constants. Define \( f = c_1f_1 + c_2f_2 \). Then

\[
\begin{align*}
I. \quad f(0) &= c_1f_1(0) + c_2f_2(0) \\
&= c_1 \int_{-1}^{1} f_1(t)dt + c_2 \int_{-1}^{1} f_2(t)dt \\
&= \int_{-1}^{1} c_1f_1(t)dt + \int_{-1}^{1} c_2f_2(t)dt \\
&= \int_{-1}^{1} f(t)dt \\
II. \quad f(0.5) &= c_1f_1(0.5) + c_2f_2(0.5) \\
&= c_1(0.5) + c_2(0) \\
&= 0
\end{align*}
\]

By Theorem 1, 4.2 E&P, Subspace Criterion, \( S \) is a subspace of \( V \).

Let \( \mathbf{c} = 2\mathbf{a} - \mathbf{b} \). Then \( \mathbf{c} \cdot \mathbf{x} = 0 \) defines \( S \). Let \( A \) be \( \mathbb{R}^n \) matrix whose rows are \( \mathbf{c}, \mathbf{0}, \mathbf{0} \). Then \( A\mathbf{x} = 0 \) defines \( S \). By Theorem 2, 4.2 E&P, \( S \) is a subspace of \( V \).

\[
C = \begin{pmatrix}
1 & 1 & -2 & 3 \\
0 & 1 & 2 & 0 \\
1 & 2 & 0 & 3 \\
1 & 3 & 2 & 3
\end{pmatrix}
\]

Frame 1. Last Frame Run[C] = \( \begin{pmatrix}
1 & 0 & -4 & 3 & 0 \\
0 & 1 & 2 & 0 & 1 \\
0 & 0 & 0 & 0 & 1
\end{pmatrix}
\)

Reduced Echelon System

\[
\begin{align*}
&x_1 - 4x_2 + 3x_4 = 0 \\
x_2 + 2x_3 = 1 \\
op = 0 \\
op = 0 \\
Free = x_3, x_4
\end{align*}
\]

Gen Sol

\[
\begin{align*}
x_1 &= 4t_1 - 3t_2 \\
x_2 &= 1 - 2t_1 \\
x_3 = t_1 \\
x_4 = t_2 \\
x = \begin{pmatrix} 0 \\ 1 \\ -2 \\ 1 \\ 0 \end{pmatrix} + t_1 \begin{pmatrix} 4 \\ 1 \\ -2 \\ 0 \end{pmatrix} + t_2 \begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix}
\end{align*}
\]

Use this page to start your solution. Staple extra pages as needed.
2. (ch5) Complete (a), (b) and either (c) or (d). Do not do both (c) and (d).

(a) [30%] Given $x''(t) + 2x'(t) + 3x(t) = 0$, which represents a damped spring-mass system with $m = 1$, $c = 2$, $k = 3$, solve the differential equation [20%] and classify the answer as over-damped, critically damped or under-damped [10%].

(b) [10%] Both undetermined coefficients and variation of parameters can solve $x'' + x' = e^{-t}$. Without actually solving, is one method faster? Explain your reasoning.

(c) [60%] Find by undetermined coefficients the steady-state periodic solution for the equation $x'' + 2x' + 5x = e^{2t}\cos(t^2)$. To save time, don’t try to evaluate integrals (it’s impossible).

(d) [60%] If you did (c) above, then skip this one! Find by variation of parameters a particular solution $x_p$ for the equation $x'' + 2x' + 5x = e^{2t}\cos(t^2)$.

\[ r^2 + 2r + 3 = 0 \]
\[ r = -1 \pm i \sqrt{2} \]

Underdamped

\[ x(t) = c_1 e^t \cos(\sqrt{2}t) + c_2 e^t \sin(\sqrt{2}t) \]

(b) Variation of parameters involves integration of exponentials. It is easy. Both are fast, but a fix-up rule slows down undetermined coefficients. I would do both of parameters first.

(c) trial $y = d_1 \cos 3t + d_2 \sin 3t$. No fix-up. Stuff into DE, solve for

\[ d_1 = -\frac{15}{26}, \quad d_2 = -\frac{5}{13} \]

Then $x_p(t)$ = steady state periodic sol

\[ = \left( -\frac{15}{26} \right) \cos 3t + \left( -\frac{5}{13} \right) \sin 3t \]

(d) $r^2 + 2r + 5 = 0$, roots $-1 \pm 2i$. $x_1 = e^t \cos 2t$, $x_2 = e^t \sin 2t$,

\[ W = \begin{vmatrix} x_1 & x_2 \\ x_1' & x_2' \end{vmatrix} = 2e^{-2t} \]

Then $x_p(t) = u_1(t)x_1(t) + u_2(t)x_2(t)$

\[ u_1 = -\int \frac{\dot{x}_2}{W} = -\int \frac{e^t \sin 2t e^{t^2} \cos(t^2) dt}{2e^{-2t}} \]

\[ u_2 = \int \frac{\dot{x}_1}{W} = \int \frac{e^t \cos 2t e^{t^2} \cos(t^2) dt}{2e^{-2t}} \]

Use this page to start your solution. Staple extra pages as needed.
3. (ch5) Complete all parts below.

(a) [75%] A non-homogeneous linear differential equation with constant coefficients has right side \( f(x) = xe^{-x} + 2x^3 - x + 5 + x \cos x \) and characteristic equation \( r^2(r+1)^3(r^2+4) = 0 \). Determine the corrected trial solution for \( y_p \) according to the method of undetermined coefficients. To save time, do not evaluate the undetermined coefficients (that is, do undetermined coefficient steps 1 and 2), but skip steps 3 and 4! Undocumented detail or guessing earns no credit.

(b) [25%] Using the *recipe* for higher order constant-coefficient differential equations, write out the general solution when the characteristic equation is \( (r^3 + 4r^2 + 2r) + r^2 = 0 \).

\[ \text{atoms of } f: \ 1, e^{-x}, x, x^2, x^3, e^{-x}, x \cos x, x \sin x \]

**trial** \( y = \text{linear combination of these atoms} \)

**corrected** \( y = (d_1 + d_2 x + d_3 x^2 + d_4 x^3) e^{-x} \)

\[ + (d_5 e^{-x} + d_6 x e^{-x}) x \]

\[ + (d_7 e^{-x} + d_8 x e^{-x}) \cos x + (d_9 e^{-x} + d_{10} x e^{-x}) \sin x \]

\[ \text{char eq roots } = 0, 0, -1, -1, -1, 2i, -2i \]

\[ L = \{ 1, x, e^{-x}, x e^{-x}, x^2 e^{-x}, \cos 2x, \sin 2x \} \]

\[ r^2(r+1)^2 r(r+2) r(r+1) = 0 \]

\[ r^2(r+1)(r+2)(r^2+4)^2 \]

\[ \text{roots: } = 0, 0, 0, 0, -1, -2, 2i, 2i, -2i, -2i \]

\[ L = \{ 1, x, x^2, x^3, e^{-x}, e^{-2x}, \cos 2x, \sin 2x, x \cos 2x, x \sin 2x \} \]

**\( y \): linear combination of atoms in \( L \)

\[ = c_1 + c_2 x + c_3 x^2 + c_4 x^3 \]

\[ + c_5 e^{-x} \]

\[ + c_6 e^{-2x} \]

\[ + (c_7 + c_8 x) \cos 2x + (c_9 + c_{10} x) \sin 2x \]

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4. (ch6) Complete all of the items below.

(a) [30%] Find the eigenvalues of the matrix \( A = \begin{bmatrix} 4 & -3 & 1 & 4 \\ 3 & 4 & -3 & 5 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 9 \end{bmatrix} \). To save time, do not find eigenvectors!

(b) [70%] Given \( A = \begin{bmatrix} 1 & 1 & -1 \\ 0 & 4 & 1 \\ 0 & 1 & 4 \end{bmatrix} \), then there exists an invertible matrix \( P \) and a diagonal matrix \( D \) such that \( AP = PD \). Which of the following is a possible column of \( P \)?

\[
\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ -1 \\ -1 \end{pmatrix}.
\]

(1) Expand \( \det(A - \lambda I) \) along row 4. Then neglect.

\[
(9 - \lambda)(9 - \lambda)(9 - \lambda)^2 + 1 = 0
\]

Roots = 9, 0, 4 ± 3i

(2) A column \( \vec{v} \) of \( P \) satisfies \( A\vec{v} = \lambda \vec{v} \) for some \( \lambda \).

\[
A \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \\ 1 \end{pmatrix} = 2 \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}
\]

\[
A \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ 5 \\ 5 \end{pmatrix} ≠ 2 \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}
\]

\[
A \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ -5 \\ 5 \end{pmatrix} = 5 \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}
\]

It works. \( \begin{pmatrix} 0 \\ -1 \end{pmatrix} \) could be a col of \( P \).
5. (ch6) Complete all parts below.

Consider the $3 \times 3$ matrix

$$A = \begin{pmatrix} 4 & 1 & -1 \\ 0 & 3 & 1 \\ 0 & 1 & 3 \end{pmatrix}.$$

Already computed are eigenpairs

$$\left(2, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}\right), \quad \left(4, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\right).$$

(a) [40%] Compute and then display an invertible matrix $P$ and a diagonal matrix $D$ such that $AP = PD$.

(c) [30%] Describe precisely, and explicitly for $A$ above, Fourier's model for the computation of $Ax$.

(c) [30%] Display the vector general solution $\mathbf{x}(t)$ of the linear differential system $\mathbf{x}' = A\mathbf{x}$.

\begin{enumerate}
\item Eigenvalues satisfy $\det(A-\lambda I) = 0$. This is $(4-\lambda)((3-\lambda)^2-1) = 0$. Roots = 4, 2, 4. We are missing eigenpair $(4, \mathbf{v})$.

$$B = A - 4I = \begin{pmatrix} 0 & 1 & -1 \\ 0 & 1 & -1 \\ 0 & 1 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 1 & -1 \\ 0 & 1 & -1 \\ 0 & 1 & -1 \end{pmatrix} \text{ combo} \quad \text{Reduced echelon sys} \quad \text{Gen sol} \quad \mathbf{x} = \begin{pmatrix} t_1 \\ 0 \\ 0 \end{pmatrix}$$

$$\text{Third eigenpair} = (4, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}). \quad P = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad D = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{pmatrix}. \quad \text{Then } AP = PD.$$

\item $A(c_1 \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + c_3 \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}) = 2c_1 \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} + 4c_2 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + 4c_3 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

\item $\mathbf{x}(t) = c_1 e^{2t} \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} + c_2 e^{4t} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + c_3 e^{0t} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$
\end{enumerate}

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