Name. KEY

2250 Midterm 2 Ver 1 [7:30, F2006]

1. **(rref)**
(a) Determine a, b such that the system has a unique solution, infinitely many solutions, or no solution:

$$\begin{array}{rcrcrcl} x &+& 2y &+& z &=& b \\ 3x &+& 5y &+& 2z &=& 2b \\ 4x &+& ay &+& 3z &=& 2+b \end{array}$$

$$\left(\begin{array}{ccc|c} 1 & 2 & 1 & b \\ 3 & 5 & 2 & 2b \\ 4 & a & 3 & 2+b \end{array}\right)$$

$$\left(\begin{array}{cccc} 1 & 2 & 1 & b \\ 0 & -1 & -1 & -b \\ 4 & a & 3 & 2+b \end{array}\right) \text{ combo}(1,2,-3)$$

$$\left(\begin{array}{cccc} 1 & 2 & 1 & b \\ 0 & 1 & 1 & b \\ 4 & a & 3 & 2+b \end{array}\right) \text{ mult}(2,-1)$$

$$\left(\begin{array}{cccc} 1 & 2 & 1 & b \\ 0 & 1 & 1 & b \\ 0 & a-8 & -1 & 2-3b \end{array}\right) \text{ combo}(1,3,-4)$$

$$\left(\begin{array}{cccc} 1 & 2 & 1 & b \\ 0 & 1 & 1 & b \\ 0 & 0 & 7-a & x \end{array}\right) \text{ combo}(2,3,8-a)$$

Let $x = 2+5b-ab$

$2-3b+b(8-a) =$
$2-3b+8b-ab =$
$2+5b-ab$

Case $7-a=0$

$$\left(\begin{array}{cccc} 1 & 2 & 1 & b \\ 0 & 1 & 1 & b \\ 0 & 0 & 0 & x \end{array}\right)$$

$x = 2+5b-ab$
$= 2-2b$ because $a=7$

$\boxed{\text{No sol if } a=7,\ b\neq 1}$

$\boxed{\infty\text{-many sols if } a=7,\ b=1 \text{ Because of one free var}}$

Case $7-a \neq 0$

$$\left(\begin{array}{cccc} 1 & 2 & 1 & b \\ 0 & 1 & 1 & b \\ 0 & 0 & 1 & x/(7-a) \end{array}\right) \text{ mult}$$

Three lead variables, Zero free var

$\boxed{\text{Unique sol for } 7-a\neq 0,\ \text{all } b}$

Use this page to start your solution. Attach extra pages as needed, then staple.

✓

Name. KEY

2. (vector spaces)

(a) [20%] Give an example of a vector space V of functions and a subspace S of V, such that S has a basis of size five. Display explicit definitions of V, S and define addition $\boxed{+}$ and scalar multiplication $\boxed{\cdot}$ on V.

(b) [40%] Let V be the vector space of all column vectors $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ and let S be the subset of V given by the equations $x_1 = 0$, $x_2 + 2x_3 = 0$. Prove that S is a subspace of V. Edwards and Penney Theorem 2, if used to shorten the proof, must be stated.

(c) [40%] Find a basis of vectors for the subspace of $\mathcal{R}^4$ given by the system of equations

$$\begin{aligned} x_1 + 2x_2 - 2x_3 + x_4 &= 0, \\ x_1 + x_2 - 3x_3 + x_4 &= 0, \\ x_2 + x_3 &= 0. \end{aligned}$$

(a) V = space of a functions on $\mathbb{R}^1$

$\boxed{+}$: $(f+g)(x) = f(x) + g(x)$, $\boxed{\cdot}$: $(c\,f)(x) = c \cdot f(x)$, x in $(-\infty, \infty)$.

S = all linear combinations of $1, x, x^2, x^3, x^4$.

Atoms are independent, so a basis $= \{1, x, x^2, x^3, x^4\}$.

(b) Define $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}$. Then the equations are just $A\vec{x} = \vec{0}$ in vector-matrix form. We apply:

<u>Thm 2</u> $S = \{\vec{x} \text{ in } \mathbb{R}^n : A\vec{x} = \vec{0}\}$ is a subspace of $\mathbb{R}^n$.

(c) $\begin{pmatrix} 1 & 2 & -2 & 1 \\ 1 & 1 & -3 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix}$ Frame 1

$\begin{pmatrix} 1 & 2 & -2 & 1 \\ 0 & -1 & -1 & 0 \\ 0 & 1 & 1 & 0 \end{pmatrix}$ Combo

$\begin{pmatrix} 1 & 2 & -2 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{pmatrix}$ mult

$\begin{pmatrix} 1 & 2 & -2 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ Combo

$\begin{pmatrix} 1 & 0 & -4 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ combo Frame 5 = rref

$$\begin{cases} x_1 - 4x_3 + x_4 = 0 \\ x_2 + x_3 = 0 \\ 0 = 0 \end{cases}$$

$x_1 = 4t_1 - t_2$
$x_2 = -t_1$
$x_3 = t_1$
$x_4 = t_2$

$$\boxed{\text{Basis} = \left\{ \begin{pmatrix} 4 \\ -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}}$$

Use this page to start your solution. Attach extra pages as needed, then staple.

Name. KEY 2250 Midterm 2 Ver 1 [7:30, F2006]

3. **(independence)** Do **only two** of the following.

(a) [50%] Let $\mathbf{u}_1 = \begin{pmatrix} 1 \\ -1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$, $\mathbf{u}_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \\ 0 \end{pmatrix}$, $\mathbf{u}_3 = \begin{pmatrix} 2 \\ 1 \\ 0 \\ 1 \\ 1 \end{pmatrix}$, $\mathbf{u}_4 = \begin{pmatrix} 1 \\ 2 \\ -1 \\ 0 \\ 0 \end{pmatrix}$. State and apply a test that decides independence or dependence of the list of four vectors.

(b) [50%] State the pivot theorem. Then extract from the list below a largest set of independent vectors.

$$\mathbf{a} = \begin{pmatrix} 1 \\ -1 \\ 0 \\ 1 \end{pmatrix}, \mathbf{b} = \begin{pmatrix} 2 \\ -2 \\ 0 \\ 2 \end{pmatrix}, \mathbf{c} = \begin{pmatrix} 3 \\ -1 \\ 0 \\ 5 \end{pmatrix}, \mathbf{d} = \begin{pmatrix} 0 \\ 2 \\ 0 \\ 2 \end{pmatrix}, \mathbf{e} = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 3 \end{pmatrix},$$

(c) [50%] [If you did (a) and (b) already, then 100% has been attained: skip this one!] Assume that 3×3 matrix D is not invertible and has rank 2. Prove:

There exist independent vectors $\mathbf{x}_1$, $\mathbf{x}_2$ such that $D\mathbf{x}_1$, $D\mathbf{x}_2$ are dependent.

(a) Test : independent $\iff$ rank$(\text{aug}(u_1, u_2, u_3, u_4)) = 4$.

$$\begin{pmatrix} 1 & 0 & 2 & 1 \\ -1 & 1 & 1 & 2 \\ 1 & 1 & 0 & -1 \\ 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \end{pmatrix} \cong \begin{pmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & 3 & 3 \\ 1 & 1 & 0 & -1 \\ 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \end{pmatrix} \cong \begin{pmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & 3 & 3 \\ 0 & 1 & -2 & -2 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & -1 & -1 \end{pmatrix} \cong \begin{pmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix} \cong \begin{pmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$\boxed{\text{Rank} = 3 \Rightarrow \text{Dependent.}}$ many combined combo, mult, swap.

(b) Pivot Theorem : The pivot columns of A are independent. Non-pivot columns of A are linear combinations of the pivot columns of A.

$$\begin{pmatrix} 1 & 2 & 3 & 0 & 1 \\ -1 & -2 & -1 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 5 & 2 & 3 \end{pmatrix} \cong \begin{pmatrix} 1 & 2 & 3 & 0 & 1 \\ 0 & 0 & 2 & 2 & 2 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 5 & 2 & 3 \end{pmatrix} \cong \begin{pmatrix} 1 & 2 & 3 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 2 & 2 \end{pmatrix} \cong \begin{pmatrix} 1 & 2 & 3 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

pivots $= 1, 3$

$\boxed{\text{Cols } 1, 3 \text{ of } A \text{ are the largest independent set of cols}}$

(c) Choose $\vec{x}_1 \neq \vec{0}$ a sol of $D\vec{x} = \vec{0}$, possible by ∞-many sols. Choose $\vec{x}_2 \neq \vec{0}$ orthogonal to $\vec{x}_1$. Then $\vec{x}_1, \vec{x}_2$ are independent. But $\{D\vec{x}_1, D\vec{x}_2\} = \{\vec{0}, D\vec{x}_2\}$, a dependent set.

Use this page to start your solution. Attach extra pages as needed, then staple.

Name. KEY

4. **(determinants and elementary matrices)**
(a) [50%] Assume given 3×3 matrices A, B. Suppose $B = E_3E_2E_1A$ and E_1, E_2, E_3 are elementary matrices representing respectively a swap, a combination and a multiply by 4. Assume $\det(A) = 5$. Find $\det(5B^2)$.
(b) [50%] Let A and B be 5×5 matrices such that $AB - A$ contains a row of ones and a row of twos. Assume $\det(B - I) = -11$. Find the value of $\det(A^2)$.

(a) $\det(5B^2) = \det(5I)\det(B)\det(B)$ by the product Thm: $\det(VW) = \det(V)\det(W)$

$\det(B) = \det(E_3)\det(E_2)\det(E_1)\det(A)$
$= (4)(-1)(1)(5)$
$= -20$

$\det(5I) = \begin{vmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{vmatrix} = 125$

$\det(5B^2) = (125)(-20)(-20)$ by line 1 above
$= (125)(400)$

(b) $\det(AB - A) = \det(A(B - I))$
$= \det(A)\det(B - I)$
$= \det(A)(-11)$

Because of proportional rows, combo implies $\det(AB - A) = 0$. Then $\det(A) = 0$, hence

$\det(A^2) = \det(AA)$
$= \det(A)\det(A)$
$= \boxed{0}$

Use this page to start your solution. Attach extra pages as needed, then staple.

Name. KEY

5. (inverses and Cramer's rule)

(a) [20%] Determine all values of x for which A^{-1} exists: $A = \begin{pmatrix} 1 & 2x-1 \\ 2 & -3 \end{pmatrix}$.

(b) [40%] Apply the adjugate formula for the inverse to find the value of the entry in row 3, column 4 of A^{-1}, given A below. Other methods are not acceptable.

$$A = \begin{pmatrix} 1 & 2 & 0 & 1 \\ -1 & 0 & -1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 2 & 0 & 2 \end{pmatrix}$$

(c) [40%] Solve for z in $A\mathbf{u} = \mathbf{b}$ by Cramer's rule. Other methods are not acceptable.

$A = \begin{pmatrix} 1 & 2 & 0 \\ 3 & 0 & 4 \\ 5 & 6 & 7 \end{pmatrix}$, $\mathbf{u} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$.

(a) $\det(A) \neq 0 \iff A^{-1}$ exists.

$\det(A) = -3 - 4x + 2$

$= -1 - 4x$

$\boxed{A^{-1} \text{ exists} \iff x \neq -1/4}$

(b) entry 3,4 of $A^{-1} = \dfrac{\operatorname{cof}(A,4,3)}{\det(A)} = \boxed{\dfrac{2}{1}}$ see below

$\det(A) = (-1)(-1)\begin{vmatrix} 1 & 2 & 1 \\ 1 & 1 & 1 \\ 1 & 2 & 2 \end{vmatrix} + (1)(1)\begin{vmatrix} 1 & 2 & 1 \\ -1 & 0 & 1 \\ 1 & 2 & 2 \end{vmatrix}$ by expansion along col 3

$= \begin{vmatrix} 1 & 2 & 0 \\ 1 & 1 & 0 \\ 1 & 2 & 1 \end{vmatrix} + \begin{vmatrix} 0 & 0 & -1 \\ -1 & 0 & 1 \\ 1 & 2 & 2 \end{vmatrix} = -1 + 2 = \boxed{1}$

$\operatorname{Cof}(A,4,3) = \begin{vmatrix} 1 & 2 & 1 \\ -1 & 0 & 1 \\ 1 & 1 & 1 \end{vmatrix}(-1)^{4+3} = -\begin{vmatrix} 0 & 2 & 1 \\ -2 & 0 & 1 \\ 0 & 1 & 1 \end{vmatrix} = (-1)^2(-2)\begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix} = \boxed{-2}$

(c) $z = \dfrac{\Delta_3}{\Delta} = \boxed{\dfrac{-12}{13}}$ see below

$\Delta = \begin{vmatrix} 1 & 2 & 0 \\ 3 & 0 & 4 \\ 5 & 6 & 7 \end{vmatrix} = \begin{vmatrix} 0 & 4 \\ 6 & 7 \end{vmatrix} - 2\begin{vmatrix} 3 & 4 \\ 5 & 7 \end{vmatrix} = -24 - 2 = -26$

$\Delta_3 = \begin{vmatrix} 1 & 2 & 1 \\ 3 & 0 & 0 \\ 5 & 6 & -1 \end{vmatrix} = (-1)(3)\begin{vmatrix} 2 & 1 \\ 6 & -1 \end{vmatrix} = -3(-8) = 24$

$\boxed{z = \dfrac{24}{-26}} = \dfrac{-12}{13}$

Use this page to start your solution. Attach extra pages as needed, then staple.