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2250 Midterm 2 Ver 3 [5:55pm, F2006]

1. (rref)
Determine a, b such that the system has a unique solution, infinitely many solutions, or no solution:

$$\begin{array}{rcrcrcl} x & + & 2y & + & z & = & 1 \\ 2x & + & 9y & + & 2z & = & -a \\ 3x & + & 3y & + & 3bz & = & 1-a \end{array}$$

$$\left(\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 2 & 9 & 2 & -a \\ 3 & 3 & 3b & 1-a \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 0 & 5 & 0 & -2-a \\ 3 & 3 & 3b & 1-a \end{array}\right) \text{ combo}(1,2,-2)$$

$$\left(\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 0 & 5 & 0 & -2-a \\ 0 & -3 & 3b-3 & -2-a \end{array}\right) \text{ combo}(1,3,-3)$$

$$\left(\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 0 & -3 & 3b-3 & -2-a \\ 0 & 5 & 0 & -2-a \end{array}\right) \text{ swap}(2,3)$$

$$\left(\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 0 & -3 & 3b-3 & -2-a \\ 0 & 0 & 5(b-1) & -\frac{8}{3}(2+a) \end{array}\right) \text{ combo}(2,3,5/3)$$

$$x = -2-a+\frac{5}{3}(-2-a) = (-6-3a-10-5a)/3 = -\frac{8}{3}(2+a)$$

Case $5(b-1)=0$

$$\left(\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 0 & -3 & 0 & -2-a \\ 0 & 0 & 0 & x \end{array}\right)$$

$\boxed{\text{No sol } \begin{cases} a \ne -2 \\ b = 1 \end{cases}}$ signal eq.

$\boxed{\infty\text{-many sol } \begin{cases} a = -2 \\ b = 1 \end{cases}}$ row of zeros!

Then 2 lead and one free var.

Case $5(b-1) \ne 0$

$$\left(\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 0 & -3 & 3(b-1) & -2-a \\ 0 & 0 & 1 & \frac{x}{5(b-1)} \end{array}\right)$$

$\boxed{\text{Unique sol } \quad b \ne 1}$

Three lead vars, zero free vars.

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2. **(vector spaces)**

(a) [25%] Give an example of a vector space V of trigonometric functions which has a proper subspace S of dimension four ($\dim(S) = 4$). Define V and S precisely, including the definition of $\boxed{+}$ and $\boxed{\cdot}$, but omit all proofs. *Proper* means *strictly smaller than.*

(b) [25%] Let V be the vector space of all column vectors $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ and let S be the subset of V given by the equations $x_1 + x_2 = 0$, $x_2 = x_1 x_3$. Prove that S is **not** a subspace of V.

(c) [50%] Find a vector basis for the subspace of $\mathcal{R}^4$ given by the system of equations

$$\begin{aligned} x_1 - x_2 + 2x_3 + 3x_4 &= 0, \\ x_1 + x_2 - 4x_3 + 3x_4 &= 0, \\ -2x_2 + 6x_3 &= 0. \end{aligned}$$

(a) Choose distinct atoms $\sin x$, $\sin 2x$, $\sin 3x$, $\sin 4x$, $\sin 5x$ and let V be the set of all linear combinations of these 5 atoms. They are independent by a Theorem, so $\dim(V) = 5$, with $\boxed{+}$: $(f+g)(x) = f(x) + g(x)$, $\boxed{\cdot}$: $(cf)(x) = c \cdot f(x)$. Define S to be all linear combinations of the first 4 atoms. By independence, $\dim(S) = 4$.

(b) Choose $\vec{x} = \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}$. Then $x_1 + x_2 = 0$ and $x_2 = x_1 x_3$. But $-\vec{x} = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$ fails $x_2 = x_1 x_3$ (it says $1 = -1$), so S is not closed under $\boxed{\cdot}$, it is not a subspace.

(c) $\begin{pmatrix} 1 & -1 & 2 & 3 \\ 1 & 1 & -4 & 3 \\ 0 & -2 & 6 & 0 \end{pmatrix}$

$\begin{pmatrix} 1 & -1 & 2 & 3 \\ 0 & 2 & -6 & 0 \\ 0 & -2 & 6 & 0 \end{pmatrix}$ combo

$\begin{pmatrix} 1 & -1 & 2 & 3 \\ 0 & 2 & -6 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ combo

$\begin{pmatrix} 1 & -1 & 2 & 3 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ mult

$\begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \end{array}$

$\begin{pmatrix} 1 & 0 & -1 & 3 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ combo

$\begin{cases} x_1 = x_3 - 3x_4 = t_1 - 3t_2 \\ x_2 = 3x_3 = 3t_1 \\ x_3 = t_1 \\ x_4 = t_2 \end{cases}$ Gen Sol $\vec{x}$.

Basis $= \left\{ \begin{pmatrix} 1 \\ 3 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -3 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}$

$\dfrac{\partial \vec{x}}{\partial t_1}$ $\dfrac{\partial \vec{x}}{\partial t_2}$

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3. (independence) Do **only two** of the following.

(a) [50%] Let $\mathbf{u} = \begin{pmatrix} 1 \\ -1 \\ 1 \\ 2 \end{pmatrix}$, $\mathbf{v} = \begin{pmatrix} 2 \\ 1 \\ 0 \\ 2 \end{pmatrix}$, $\mathbf{w} = \begin{pmatrix} 1 \\ 2 \\ -1 \\ 0 \end{pmatrix}$. State a test [10%] and apply it [40%] to decide independence or dependence of the list of vectors $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$.

(b) [50%] State the pivot theorem [10%], then extract from the list below a largest set of independent vectors [40%].

$$\mathbf{a} = \begin{pmatrix} 1 \\ -1 \\ 0 \\ -1 \end{pmatrix}, \mathbf{b} = \begin{pmatrix} 2 \\ -2 \\ 0 \\ -2 \end{pmatrix}, \mathbf{c} = \begin{pmatrix} 2 \\ 0 \\ 0 \\ 2 \end{pmatrix}, \mathbf{d} = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 3 \end{pmatrix}, \mathbf{e} = \begin{pmatrix} 5 \\ -1 \\ 0 \\ 3 \end{pmatrix}.$$

(c) [50%] [If you did the two previous problems, then 100% has been obtained: skip this one!] Let $D = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 0 & 1 \\ 3 & 2 & 5 \end{pmatrix}$. Prove:

There exist independent vectors $\mathbf{x}_1$, $\mathbf{x}_2$ such that $D\mathbf{x}_1$, $D\mathbf{x}_2$ is a dependent set.

(a) Test: independent $\iff$ rank$(\text{aug}(\vec{u},\vec{v},\vec{w})) = 3$.

$$\begin{pmatrix} 1 & 2 & 1 \\ -1 & 1 & 2 \\ 1 & 0 & -1 \\ 2 & 2 & 0 \end{pmatrix} \quad \begin{pmatrix} 1 & 2 & 1 \\ 0 & 3 & 3 \\ 1 & 0 & -1 \\ 2 & 2 & 0 \end{pmatrix} \nearrow \begin{pmatrix} 1 & 2 & 1 \\ 0 & 3 & 3 \\ 0 & -2 & -2 \\ 0 & -2 & -2 \end{pmatrix} \quad \begin{pmatrix} 1 & 2 & 1 \\ 0 & 3 & 3 \\ 0 & -2 & -2 \\ 2 & 2 & 0 \end{pmatrix} \nearrow \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$\boxed{\text{rank} = 2 \quad \text{Dependent}}$

another way, faster: $W = V - U$, so cols are dependent, rank < 3.

(b) Pivot Theorem: The pivot columns of A are independent. Any other column is a linear combination of the pivot columns.

$$\begin{pmatrix} 1 & 2 & 2 & 1 & 5 \\ -1 & -2 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ -1 & -2 & 2 & 3 & 3 \end{pmatrix} \cong \begin{pmatrix} 1 & 2 & 2 & 1 & 5 \\ 0 & 0 & 2 & 2 & 4 \\ 0 & 0 & 0 & 0 & 0 \\ -1 & -2 & 2 & 3 & 3 \end{pmatrix} \cong \begin{pmatrix} 1 & 2 & 2 & 1 & 5 \\ -1 & -2 & 2 & 3 & 3 \\ 0 & 0 & 2 & 2 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \cong \begin{pmatrix} 1 & 2 & 2 & 1 & 5 \\ 0 & 0 & 4 & 4 & 8 \\ 0 & 0 & 2 & 2 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

pivots = cols 1, 3

$\boxed{\text{Vectors } \vec{a}, \vec{c}}$

(c) $\det(D) = \begin{vmatrix} 1 & 2 & 3 \\ 1 & 0 & 1 \\ 3 & 2 & 5 \end{vmatrix} = -\begin{vmatrix} 2 & 3 \\ 2 & 5 \end{vmatrix} - \begin{vmatrix} 1 & 2 \\ 3 & 2 \end{vmatrix} = -(10-6) - (2-6) = 0$.

$D\vec{x} = \vec{0}$ has ∞-many sols. Choose $\vec{x}_1 \neq \vec{0}$ with $D\vec{x}_1 = \vec{0}$. Choose $\vec{x}_2 \neq \vec{0}$ with $\vec{x}_1 \perp \vec{x}_2$. Then $\vec{x}_1, \vec{x}_2$ are independent. But $\{D\vec{x}_1, D\vec{x}_2\} = \{0, D\vec{x}_2\}$, so it is a dependent set.

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4. (determinants and elementary matrices)

(a) [40%] Assume given 3×3 matrices A, B. Suppose $B^2 = E_4 E_3 E_2 E_1 A$ and E_1, E_2, E_3, E_4 are elementary matrices representing respectively a swap, a combination, a swap and a multiply by 5. Assume $\det(B) = -2$. Compute the value of $\det(-A^2)$.

(b) [20%] Let A be a 2×2 matrix such that $A\mathbf{x} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$ has infinitely many solutions. Prove that A does not have an inverse.

(c) [40%] Let $A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 7 \\ 0 & 0 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$. Then $B = \mathbf{rref}(A)$. Find explicit elementary matrices $E_1, E_2, \ldots, E_k$ such that $\mathbf{rref}(A) = E_k \cdots E_2 E_1 A$. Likely $k = 3$, if you are efficient.

ⓐ
$$\det(-A^2) = \det((-I)A^2) = \det(-I)\det(A)\det(A) = (-1)^3(\det(A))^2$$
$$\det(B^2) = \det(E_4)\det(E_3)\det(E_2)\det(E_1)\det(A)$$
$$(\det(B))^2 = (5)(-1)(1)(-1)\det(A)$$
$$(-2)^2 = 5\det(A)$$
$$\boxed{\det(-A^2) = (-1)^3\left(\frac{(-2)^2}{5}\right)^2 = \frac{-16}{25}}$$

ⓑ To have ∞-many solutions, there must be one free variable, so $\text{rank}(A) < 2$. Then $\det(A) = 0$ and A^{-1} does not exist.

ⓒ
$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 7 \\ 0 & 0 & 1 \end{pmatrix}$$
$$\begin{pmatrix} 1 & 2 & 3 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix} \text{combo}(1,2,-2) \qquad E_1 = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$
$$\begin{pmatrix} 1 & 2 & 3 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \text{combo}(2,3,-1) \qquad E_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix}$$
$$\begin{pmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \text{combo}(2,1,-3) \qquad E_3 = \begin{pmatrix} 1 & -3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$
$$\text{rref}(A) = E_3 E_2 E_1 A$$
$$\boxed{B = \begin{pmatrix} 1 & -3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} A}$$

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5. **(inverses and Cramer's rule)**

(a) [20%] Given $A = \begin{pmatrix} 1 & 2x-1 \\ -2 & 3 \end{pmatrix}$, assume $A\mathbf{x} = \mathbf{b}$ has a unique solution for all vectors $\mathbf{b}$. Find the possible values of x.

(b) [50%] Apply the adjugate formula for the inverse [no credit for **rref** methods] to find the value of the entry in row 2, column 3 of A^{-1}, given

$$A = \begin{pmatrix} 1 & -1 & 1 & 1 \\ 2 & 0 & 1 & 2 \\ 0 & -1 & 1 & 0 \\ 1 & 1 & 1 & 2 \end{pmatrix}$$

(c) [30%] Solve for z in $A\mathbf{u} = \mathbf{b}$ by Cramer's rule [**rref** methods not acceptable]:

$$A = \begin{pmatrix} 1 & 2 & 0 \\ 3 & 0 & 1 \\ 5 & 0 & 1 \end{pmatrix}, \quad \mathbf{u} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}.$$

(a) $A\vec{x} = \vec{b}$ has a unique sol for all $\vec{b} \iff A^{-1}$ invertible $\iff \det(A) \neq 0$.

$\det(A) = 3 + 4x - 2 = 1 + 4x$. $\boxed{\text{ans: } 1+4x \neq 0}$

(b) Entry in row 2, col 3 of $A^{-1} = \dfrac{\mathrm{cof}(A,3,2)}{\det(A)} = \boxed{\dfrac{1}{1}}$ see below

$\mathrm{cof}(A,3,2) = (-1)^{3+2} \begin{vmatrix} 1 & 1 & 1 \\ 2 & 1 & 2 \\ 1 & 1 & 2 \end{vmatrix} = (-1) \begin{vmatrix} 1 & 1 & 1 \\ 2 & 1 & 2 \\ 0 & 0 & 1 \end{vmatrix}$ combo $= (-1)\begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix} = \boxed{1}$

$\det(A) = \begin{vmatrix} 1 & -1 & 1 & 0 \\ 2 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{vmatrix} = \begin{vmatrix} 1 & -1 & 1 \\ 2 & 0 & 1 \\ 0 & -1 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 2 & 0 & 1 \\ 0 & -1 & 1 \end{vmatrix}$ combo $= \boxed{1}$

← combo col 4

(c) $\boxed{z = \dfrac{\Delta_3}{\Delta} = \dfrac{-16}{4} = -4}$ see below

$\Delta = \begin{vmatrix} 1 & 2 & 0 \\ 3 & 0 & 1 \\ 5 & 0 & 1 \end{vmatrix} = -2 \begin{vmatrix} 3 & 1 \\ 5 & 1 \end{vmatrix} = \boxed{4}$

$\Delta_3 = \begin{vmatrix} 1 & 2 & 2 \\ 3 & 0 & -1 \\ 5 & 0 & 1 \end{vmatrix} = -2 \begin{vmatrix} 3 & -1 \\ 5 & 1 \end{vmatrix} = \boxed{-16}$

↑ $\vec{b}$ in col 3

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