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1. (rref)
Determine a, b such that the system has a unique solution, infinitely many solutions, or no solution:

$$\begin{array}{rcrcrcl} x & + & 2y & + & z & = & a \\ 5x & + & 9y & + & 2z & = & 3-3a \\ 6x & + & 3y & + & 3bz & = & 2-a \end{array}$$

$$\left(\begin{array}{ccc|c} 1 & 2 & 1 & a \\ 5 & 9 & 2 & 3-3a \\ 6 & 3 & 3b & 2-a \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 1 & 2 & 1 & a \\ 0 & -1 & -3 & 3-8a \\ 6 & 3 & 3b & 2-a \end{array}\right) \quad \text{combo}(1,2,-5)$$

$$\left(\begin{array}{ccc|c} 1 & 2 & 1 & a \\ 0 & -1 & -3 & 3-8a \\ 0 & -9 & 3b-6 & 2-7a \end{array}\right) \quad \text{combo}(1,3,-6)$$

$$\left(\begin{array}{ccc|c} 1 & 2 & 1 & a \\ 0 & -1 & -3 & 3-8a \\ 0 & 0 & 3b+21 & x \end{array}\right) \quad \text{combo}(2,3,-9)$$

$$\begin{aligned} x &= 2-7a-9(3-8a) \\ &= -25-7a+72a \\ &= -25+65a \end{aligned}$$

Case $3b+21=0$

Then at most 2 lead vars.

$$\left(\begin{array}{ccc|c} 1 & 2 & 1 & a \\ 0 & -1 & -3 & 3-8a \\ 0 & 0 & 0 & x \end{array}\right)$$

$\boxed{\text{No sol for } -25+65a \neq 0 \text{ and } b=-7}$

$\boxed{\infty\text{-many sols for } -25+65a=0 \text{ and } b=-7}$

Case $3b+21 \neq 0$

Then there are 3 lead vars and zero free vars.

$$\left(\begin{array}{ccc|c} 1 & 2 & 1 & a \\ 0 & -1 & -3 & 3-8a \\ 0 & 0 & 1 & x/(3b-21) \end{array}\right)$$

$\boxed{\text{unique sol for } b \neq -7 \text{ and all values of } a}$

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2. (vector spaces)

(a) [25%] Give an example of a vector space V of matrices which has a proper subspace S of dimension five ($\dim(S) = 5$). Define V and S precisely, but omit all proofs. *Proper* means *strictly smaller than.*

(b) [25%] Let V be the vector space of all column vectors $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ and let S be the subset of V given by the equations $x_1 + x_2 = 0$, $x_2 = x_3$. Prove that S is a subspace of V. If you cite Edwards and Penney Theorem 2, in order to shorten details, then please state Theorem 2.

(c) [50%] Find a vector basis for the subspace of $\mathcal{R}^4$ given by the system of equations

$$\begin{aligned} x_1 - x_2 + 2x_3 + x_4 &= 0, \\ x_1 + x_2 - 3x_3 + x_4 &= 0, \\ -2x_2 + 5x_3 &= 0. \end{aligned}$$

(a) V = data set of all 3×3 matrices with $\boxed{+}$ and $\boxed{\cdot}$ defined component wise.

S = subset of all linear combinations of

$$\begin{pmatrix} 1&0&0\\0&0&0\\0&0&0 \end{pmatrix}, \begin{pmatrix} 0&1&0\\0&0&0\\0&0&0 \end{pmatrix}, \begin{pmatrix} 0&0&1\\0&0&0\\0&0&0 \end{pmatrix}, \begin{pmatrix} 0&0&0\\1&0&0\\0&0&0 \end{pmatrix}, \begin{pmatrix} 0&0&0\\0&1&0\\0&0&0 \end{pmatrix}.$$

Motivated by BMP format data storage of image sensor data in a digital camera. The set S is just the first five pixels of the photo.

(b) Let $A = \begin{pmatrix} 1&1&0\\0&1&-1\\0&0&0 \end{pmatrix}$. Then the equations in matrix vector form are $A\vec{x} = \vec{0}$. Apply

<u>Theorem 2</u>: The set $S = \{\vec{x} \text{ in } \mathbb{R}^3 : A\vec{x} = \vec{0}\}$ is a subspace of $\mathbb{R}^3$.

(c)

$$\begin{pmatrix} 1&-1&2&1\\1&1&-3&1\\0&-2&5&0 \end{pmatrix} \to \begin{pmatrix} 1&-1&2&1\\0&2&-5&0\\0&-2&5&0 \end{pmatrix} \to \begin{pmatrix} 1&-1&2&1\\0&2&-5&0\\0&0&0&0 \end{pmatrix}$$

$$\to \begin{pmatrix} 1&0&-1/2&1\\0&2&-5&0\\0&0&0&0 \end{pmatrix} \to \begin{pmatrix} 1&0&-1/2&1\\0&1&-5/2&0\\0&0&0&0 \end{pmatrix} \to \begin{cases} x_1 - x_3/2 + x_4 = 0 \\ x_2 - 5x_3/2 = 0 \\ 0 = 0 \end{cases}$$

$x_1 = t_1/2 - t_2$
$x_2 = 5t_1/2$
$x_3 = t_1$
$x_4 = t_2$

$$\boxed{\text{Basis} = \left\{ \begin{pmatrix} 1/2\\5/2\\1\\0 \end{pmatrix}, \begin{pmatrix} -1\\0\\0\\1 \end{pmatrix} \right\}}$$

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3. **(independence)** Do **only two** of the following.

(a) [50%] Let $\mathbf{u} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$, $\mathbf{v} = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$, $\mathbf{w} = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$. State a test [10%] and apply it [40%] to decide independence or dependence of the list of vectors **u**, **v**, **w**.

(b) [50%] State the pivot theorem [10%], then extract from the list below a largest set of independent vectors [40%].

$$\mathbf{a} = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 3 \end{pmatrix}, \mathbf{b} = \begin{pmatrix} 2 \\ -2 \\ 0 \\ -2 \end{pmatrix}, \mathbf{c} = \begin{pmatrix} 1 \\ -1 \\ 0 \\ -1 \end{pmatrix}, \mathbf{d} = \begin{pmatrix} 5 \\ -1 \\ 0 \\ 3 \end{pmatrix}, \mathbf{e} = \begin{pmatrix} 2 \\ 0 \\ 0 \\ 2 \end{pmatrix},$$

(c) [50%] [If you did the two previous problems, then 100% has been obtained: skip this one!] Assume that a 3×3 matrix D has a row of zeros. Prove:

There exist independent vectors $\mathbf{x}_1$, $\mathbf{x}_2$ such that $D\mathbf{x}_1$, $D\mathbf{x}_2$ is a dependent set.

(a) Test: indep $\iff$ determinant of aug$(\vec{u}, \vec{v}, \vec{w})$ is not zero.

$$\begin{vmatrix} 1 & 2 & 1 \\ -1 & 1 & 2 \\ 1 & 0 & -1 \end{vmatrix} = (1)(1)\begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} + 0 + (1)(-1)\begin{vmatrix} 1 & 2 \\ -1 & 1 \end{vmatrix}$$

Cofactor expansion along row 3

$$= 3 - 3$$
$$= 0$$

$\boxed{\text{Dependent}}$

(b) Pivot Theorem: The pivot columns of A are independent. The other columns of A are linear combinations of the pivot columns.

$$\begin{pmatrix} 1 & 2 & 1 & 5 & 2 \\ 1 & -2 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 3 & -2 & -1 & 3 & 2 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 1 & 5 & 2 \\ 0 & -4 & -2 & -6 & -2 \\ 0 & 0 & 0 & 0 & 0 \\ 3 & -2 & -1 & 3 & 2 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 1 & 5 & 2 \\ 0 & 4 & 2 & 6 & 2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & -8 & -4 & -12 & -4 \end{pmatrix}$$

$$\to \begin{pmatrix} 1 & 2 & 1 & 5 & 2 \\ 0 & 4 & 2 & 6 & 2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

pivots = col 1, col 2.

$\boxed{\vec{a}, \vec{b} \text{ are a largest indep set of cols of } A}$

(c) Choose $\vec{x}_1 \neq \vec{0}$ a solution of $D\vec{x} = \vec{0}$, possible because rank$(D) < 3$, which means one free vars. Choose $\vec{x}_2 \neq \vec{0}$ orthogonal to $\vec{x}_1$. Then $\vec{x}_1, \vec{x}_2$ are independent, but $\{D\vec{x}_1, D\vec{x}_2\} = \{\vec{0}, D\vec{x}_2\}$ is a dependent set.

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4. **(determinants and elementary matrices)**
(a) [50%] Assume given 4×4 matrices A, B. Suppose $B = E_4E_3E_2E_1A$ and E_1, E_2, E_3, E_4 are elementary matrices representing respectively a swap, a combination, a swap and a multiply by 3/2. Assume $\det(B) = -1/2$. Compute the value of $\det(4A^3)$.
(b) [50%] Let A be a 5×5 matrix such that $A\mathbf{x} = E\mathbf{b}$ has a unique solution for all column vectors $\mathbf{b}$ and all elementary matrices E. Prove that A has an inverse [25%] and solve for $\mathbf{x}$ [25%] in terms of A, E and $\mathbf{b}$.

(a)
$$\det(4A^3) = \det((4I)(A^3)) = \det(4I)\det(A^3) = 4^4(\det(A))^3$$

by the product rule and the triangular rule.

$$\det(B) = \det(E_4E_3E_2E_1A) = \det(E_4)\det(E_3)\det(E_2)\det(E_1)\det(A)$$
$$-1/2 = (3/2)(-1)(1)(-1)\det(A)$$

use given and results for el. mat.

$$\det(4A^3) = 256(\det A)^3 = 256\left(-\frac{1}{2}\cdot\frac{2}{3}\right)^3 = \boxed{-\frac{256}{27}}$$

(b) • Choose $\vec{b} = \vec{0}$ to show $A\vec{x} = \vec{0}$ has a unique solution $\vec{x} = \vec{0}$. Then $\mathrm{rref}(A) = I$ and A has an inverse.

• To solve, multiply $A\vec{x} = E\vec{b}$ by A^{-1}:

$$A^{-1}A\vec{x} = A^{-1}E\vec{b}$$
$$I\vec{x} = A^{-1}E\vec{b}$$
$$\boxed{\vec{x} = A^{-1}E\vec{b}}$$

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5. (inverses and Cramer's rule)

(a) [20%] Determine all values of x for which A^{-1} exists: $A = \begin{pmatrix} 1 & 2x-1 \\ -2 & 3 \end{pmatrix}$.

(b) [50%] Apply the adjugate formula for the inverse to find the value of the entry in row 1, column 4 of A^{-1}, given

$$A = \begin{pmatrix} 1 & 2 & 0 & 1 \\ -1 & 0 & -1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 2 & 0 & 2 \end{pmatrix}$$

(c) [30%] Solve for x in $A\mathbf{u} = \mathbf{b}$ by Cramer's rule [**rref** methods not acceptable]:

$$A = \begin{pmatrix} 1 & 2 & 0 \\ 3 & 0 & 1 \\ 5 & 0 & 1 \end{pmatrix}, \quad \mathbf{u} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}.$$

(a) A^{-1} exists $\Leftrightarrow$ $\det(A) \neq 0$

$$\det(A) = \begin{vmatrix} 1 & 2x-1 \\ -2 & 3 \end{vmatrix} = 3 + 4x - 2 = 1 + 4x$$

$$\boxed{A^{-1} \text{ exists} \Leftrightarrow x \neq -1/4}$$

(b) entry in row 1 column 4 $= \dfrac{\mathrm{Cof}(A,4,1)}{\det(A)} = \boxed{\dfrac{3}{1}}$ see below

$$\mathrm{Minor}(A,4,1) = \begin{vmatrix} 2 & 0 & 1 \\ 0 & -1 & 1 \\ 1 & 1 & 1 \end{vmatrix} = 2\begin{vmatrix} -1 & 1 \\ 1 & 1 \end{vmatrix} + 0 + \begin{vmatrix} 0 & -1 \\ 1 & 1 \end{vmatrix} = -4 + 1 = -3$$

$\mathrm{Cof}(A,4,1) = \boxed{3}$

$$\det(A) = (-1)(-1)\begin{vmatrix} 1 & 2 & 1 \\ 1 & 1 & 1 \\ 1 & 2 & 2 \end{vmatrix} + (1)(1)\begin{vmatrix} 1 & 2 & 1 \\ -1 & 0 & 1 \\ 1 & 2 & 2 \end{vmatrix}$$ by Col 3 cofactor exp.

$= -1 + 2 = \boxed{1}$ by visual combo + cofactor expansion

(c) $x = \dfrac{\Delta_1}{\Delta} = \dfrac{-2}{4} = \boxed{-\dfrac{1}{2}}$ see below

$$\Delta = \begin{vmatrix} 1 & 2 & 0 \\ 3 & 0 & 1 \\ 5 & 0 & 1 \end{vmatrix} = (-1)(2)\begin{vmatrix} 3 & 1 \\ 5 & 1 \end{vmatrix} = 4$$

$$\Delta_1 = \begin{vmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 1 \end{vmatrix} = (-1)(2)\begin{vmatrix} 0 & 1 \\ -1 & 1 \end{vmatrix} = -2$$

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