Name. KEY Time of your class ________

Differential Equations and Linear Algebra 2250 [7:30] Version 1

Midterm Exam 1

Tuesday, 3 October 2006

Instructions: This in-class exam is 50 minutes. No calculators, notes, tables or books. No answer check is expected. Details count 75%. The answer counts 25%.

1. **(Quadrature Equation)**
   Solve for the general solution $y(x)$ in the equation $y' = xe^{-x} + 2\sec^2 x + \tan^2 x + \dfrac{x^3}{4+x^2}$.

$\int y'\,dx = \int F\,dx \qquad F = F_1 + F_2 + F_3$

$F_1 = xe^{-x}, \quad \int F_1\,dx = -(x+1)e^{-x} + c_1$ problem 1.2-10

$F_2 = 2\sec^2 x + \tan^2 x$

$\quad = 3\sec^2 x - 1$

use $\cos^2 x + \sin^2 x = 1$, divide by $\cos^2 x$.

$\int F_2\,dx = 3\tan x - x + c_2$

$F_3 = \dfrac{x^3}{4+x^2}$

$\quad = x + \dfrac{-4x}{4+x^2}$

$$\begin{array}{r} x \\ 4+x^2\,\overline{)\,x^3} \\ \underline{x^3+4x} \\ -4x \end{array}$$

$\int F_3\,dx = \dfrac{x^2}{2} + \int \dfrac{-4x\,dx}{4+x^2}$

$u = 4+x^2$

$du = 2x\,dx$

$\quad = \dfrac{x^2}{2} + \int \dfrac{-2\,du}{u}$

$\quad = \dfrac{x^2}{2} + (-2)\ln|u| + c_3$

$\quad = \dfrac{x^2}{2} - 2\ln|4+x^2| + c_3$

$y = \int F\,dx$

$\quad = \int (F_1 + F_2 + F_3)\,dx$

$\quad = 3\tan x - x \leftarrow$ Fix

$y = -(x+1)e^{-x} + 3\tan x - x + \dfrac{x^2}{2} - 2\ln(4+x^2) + c$

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2. **(Classification of Equations)**
The problem $y' = f(x,y)$ is defined to be **separable** provided $f(x,y) = F(x)G(y)$ for some functions F and G.

(a) [60%] Check ($\boxed{\text{X}}$) the problems that can be put into separable form, but don't supply any details.

☒	$y' = -y(xy+1)+(x+1)y$ $= -xy^2 \cancel{-y} + xy \cancel{+y}$	☒	$yy' = xy^2 + x$
☐	$y' = xe^y + ye^x$	☒	$y' + y = e^2$

(b) [15%] State a test which can verify that an equation $y' = f(x,y)$ is not linear.

(c) [25%] Use the separable equation test to show that $y' = e^x + y$ is not separable.

(b) The equation is linear provided $\frac{\partial f}{\partial y}$ is independent of y.
The equation is not linear provided $\frac{\partial f}{\partial y}$ depends on y.

(c) Let $f(x,y) = e^x + y$

Choose $x_0 = 0$, $y_0 = 0$. $f(0,0) = e^0 + 0 = 1 \neq 0$

Define $F(x) = \frac{f(x,y_0)}{f(x_0,y_0)} = \frac{f(x,0)}{1} = e^x$

Define $G(y) = f(x_0,y) = e^0 + y = 1 + y$

$$\begin{aligned} FG &= e^x(1+y) \\ &= e^x + ye^x \\ &\neq e^x + y = f \end{aligned}$$

By the Test, $y' = e^x + y$ is not separable.

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3. (Solve a Separable Equation)

Given $y^3y' = \left(\sec x \tan x + \left(\frac{x}{1+x}\right)^2\right)(5-y)$.

(a) Find all equilibrium solutions.

(b) Find the non-equilibrium solution in implicit form.

To save time, **do not solve** for y explicitly.

(a) $\boxed{y=5}$ set $y=c$, solve for $c=5$. Equil sol $y=5$.

(b) $$\int \frac{y^3y'dx}{5-y} = \int \sec x \tan x \, dx + \int \left(\frac{x}{1+x}\right)^2 dx$$

$$\int \frac{u^3du}{5-u} = \sec x + \int \left(\frac{x}{1+x}\right)^2 dx \qquad \begin{cases} u=y(x) \\ du=y'dx \end{cases}$$

Apply quadr method after in form $\frac{y'}{G(y)} = F(x)$

$$\text{LHS} = \int \left(-u^2-5u-25+\frac{125}{5-u}\right)du \qquad \text{because} \rightarrow$$

$$\begin{array}{r|l} & -u^2-5u-25 \\ \hline 5-u & u^3 \\ & \underline{u^3-5u^2} \\ & 5u^2 \\ & \underline{5u^2-25u} \\ & 25u \\ & \underline{25u-125} \\ & 125 \end{array}$$

$$= -\frac{u^3}{3} - \frac{5u^2}{2} - 25u - 125\ln|5-u| + c_1$$

$$\text{RHS} = \sec x + \int \left(\frac{x}{1+x}\right)^2 dx$$

$$= \sec x + \int \left(\frac{w-1}{w}\right)^2 dw \qquad \begin{cases} w=1+x \\ dw=dx \end{cases}$$

$$= \sec x + \int \left(1-\frac{2}{w}+\frac{1}{w^2}\right)dw$$

$$= \sec x + w - 2\ln|w| - w^{-1} + c_2$$

$$= \sec x + (1+x) - 2\ln|1+x| - \frac{1}{1+x} + c_2$$

ans:

$$-\frac{y^3}{3} - \frac{5y^2}{2} - 25y - 125\ln|5-y| = \sec x + \underset{\substack{\uparrow \\ \text{or } 1+x}}{x} - 2\ln|1+x| - \frac{1}{1+x} + c$$

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4. (Linear Equations)

(a) [60%] Solve $x'(t) = -16 + \frac{1}{2t+1}x(t)$, $x(0) = -16$. Show all integrating factor steps.

(b) [30%] Solve the homogeneous equation $\sqrt{2x+3}\,\frac{dy}{dx} = y$. The answer contains symbol c.

(c) [10%] The problem $2\sqrt{2x+3}\,y' = 2y + 7$ can be solved using the answer y_h from (b) plus superposition $y = y_h + y_p$. Give an equation for y_p.

(a) $x' + \frac{-1}{2t+1}x = -16$

$$\frac{(Wx)'}{W} = -16$$

$$W = e^{\int \frac{-dt}{2t+1}} = e^{\frac{-1}{2}\ln|2t+1|} = e^{\ln(2t+1)^{-1/2}} = (2t+1)^{-1/2}$$

near $t=0$

$$(Wx)' = -16W$$

$$\int (Wx)'dt = -16\int W\,dt$$

$$Wx = -16\frac{(2t+1)^{1/2}}{1/2}\cdot\frac{1}{2} + c$$

$$x = -16(2t+1) + c(2t+1)^{1/2}$$

$$-16 = -16(0+1) + c$$

at $t=0$

$$\boxed{x = -16(2t+1)}$$

(b) $y' - \frac{1}{\sqrt{2x+3}}y = 0$

$$\frac{(Wy)'}{W} = 0$$

$$(Wy)' = 0$$

$$y = cW^{-1}$$

$$\boxed{y = ce^{\sqrt{2x+3}}}$$

$$W = e^{-\int \frac{dx}{(2x+3)^{1/2}}} = e^{-\int (2x+3)^{-1/2}dx} = e^{-(2x+3)^{1/2}\cdot\frac{1}{1/2}\frac{1}{2}} = e^{-\sqrt{2x+3}}$$

Eq. only makes sense for $2x+3>0$

(c) $\boxed{y_p = -7/2}$ It's a constant or equilibrium solution found by setting $y=c$, then solve for c.

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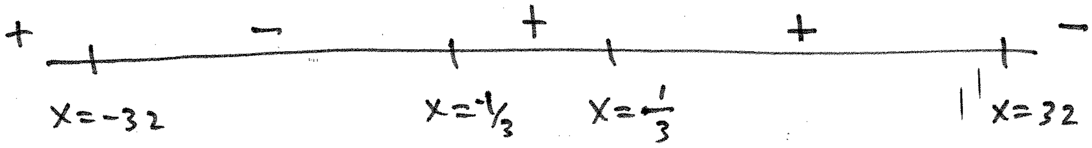
5. (Stability)
(a) [50%] Draw a phase line diagram for the differential equation

$$dx/dt = \left(2 - \sqrt[5]{|x|}\right)^3 (1-3x)(1-9x^2)(36x^2-4)^4.$$

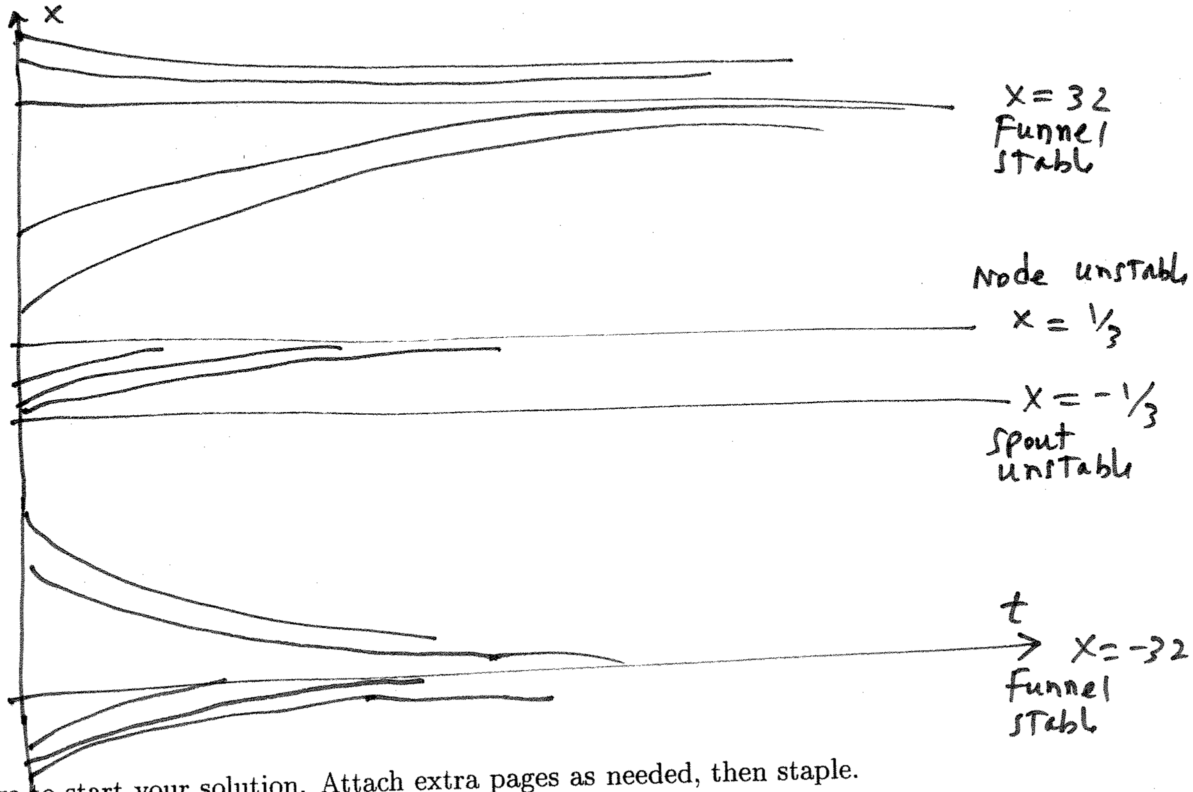
Expected in the diagram are equilibrium points and signs of x' (or flow direction markers $<$ and $>$).
(b) [50%] Draw a phase diagram using the phase line diagram of (a). Add these labels as appropriate: funnel, spout, node, source, sink, stable, unstable. Show at least 8 threaded curves. A direction field is not expected or required.

(a) roots: $x = 32$, $x = -32$, $x = \frac{1}{3}$, $x = -\frac{1}{3}$

$$f(x) = \left(2 - |x|^{1/5}\right)^3 (1-3x)^6 (1+3x)^5 (4^4)$$



(b)



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