Name. KEY Time of your class ________

Differential Equations and Linear Algebra 2250 [5:55pm] Version 3

Midterm Exam 1

Wednesday, 4 October 2006

Instructions: This in-class exam is 50 minutes. No calculators, notes, tables or books. No answer check is expected. Details count 75%. The answer counts 25%.

1. (Quadrature Equation)
Solve for the general solution $y(x)$, $x > 0$, in the equation

$$y' = xe^{x^2}\sec^2 x - xe^{x^2}\tan^2 x + \sin x\cos x + \frac{x^3}{2+x}.$$

$$y' = xe^{x^2}(\sec^2 x - \tan^2 x) + \sin x\cos x + \frac{x^3}{2+x}$$

$$y' = xe^{x^2}(1) + (\sin x)\cos x + x^2 - 2x + 4 + \frac{-8}{2+x}$$

$$\int y'dx = \int xe^{x^2}dx + \int (\sin x)\cos x\,dx + \int (x^2-2x+4)dx + \int \frac{-8dx}{2+x}$$

$$\boxed{y = \frac{1}{2}e^{x^2} + \frac{\sin^2 x}{2} + \frac{x^3}{3} - x^2 + 4x - 8\ln|2+x| + c}$$

$$\begin{array}{r} x^2 - 2x + 4 \\ x+2 \,\big)\, x^3 \\ x^3 + 2x^2 \\ \hline -2x^2 \\ -2x^2 - 4x \\ \hline 4x \\ 4x + 8 \\ \hline -8 \end{array}$$

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2. (Classification of Equations)
The problem $y' = f(x, y)$ is defined to be **separable** provided $f(x, y) = F(x)G(y)$ for some functions F and G.

(a) [60%] Check ($\boxed{X}$) the problems that can be put into separable form, but don't supply any details.

$\boxtimes$	$y' = -y(2y + x) + (x + 1)y$ $= -2y^2 - yx + xy + y$	$\square$	$e^y y' = xy^2 + x^2$
$\boxtimes$	$y' = (\sin(x)e^{x+y})^2$	$\square$	$y' + \pi^2 y = e^{\pi x}$

(b) [15%] State a test which can verify that an equation $y' = f(x, y)$ is linear but not quadrature.

(c) [25%] Use the separable equation test to show that $y' = \sin(xy)$ is not separable.

(b) $\frac{\partial f}{\partial y}$ nonzero and independent of y
or
$f(x,y) = q(x) - p(x)y$ with $p(x) \neq 0$

(c) Choose $x_0 = 1$, $y_0 = \pi/2$, Then $f(1, \pi/2) = \sin(\frac{\pi}{2}) = 1 \neq 0$.

Define $F = \frac{f(x, \pi/2)}{f(1, \pi/2)} = \sin(\pi x/2)$

$G = f(1, y) = \sin(y)$

Then $FG = \sin\left(\frac{\pi x}{2}\right)\sin(y)$

$\neq \sin(xy) = f$

(because, otherwise, for $x = 2$ and $y = \pi/4$ we get $0 = 1$)

Therefore, $y' = \sin(xy)$ is not separable.

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3. (Solve a Separable Equation)
Given $y' = \left(\dfrac{\cos x}{\sin^2 x} + \dfrac{x}{1+x}\right)\tan^2(1+y)$.
(a) Find all equilibrium solutions.
(b) Find the non-equilibrium solution in implicit form.
To save time, **do not solve** for y explicitly. Hint: change the tangent to sines and cosines and use $\cos^2\theta + \sin^2\theta = 1$.

(a) For $y' = F(x)G(y)$, we solve $G(c)=0$ for c and report all answers $y=c$. Then $\mathrm{Tan}^2(1+c)=0$, or $1+c = n\pi$. ans: $\boxed{y = n\pi - 1 \text{ for } n = 0, \pm 1, \pm 2, \ldots}$

(b) Start with $\dfrac{y'}{G(y)} = F(x)$ and apply quadrature as follows.

$$\int \frac{y'dx}{\mathrm{Tan}^2(1+y)} = \int \sin^{-2}(x)\cos(x)dx + \int \left(1 - \frac{1}{1+x}\right)dx$$
$$= -\frac{1}{\sin x} + x - \ln|1+x| + c$$
$$= -\csc x + x - \ln|1+x| + c$$

Because $\dfrac{1}{\mathrm{Tan}^2\theta} = \dfrac{\cos^2\theta}{\sin^2\theta} = \dfrac{1-\sin^2\theta}{\sin^2\theta} = \csc^2\theta - 1$
it follows that

$$\boxed{-y - \cot(1+y) = x - \csc x - \ln|1+x| + c}$$

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4. (Linear Equations)

(a) [60%] Solve $y' = -16 + \dfrac{1}{3x+1}y$, $y(0) = -8$. Show all integrating factor steps.

(b) [20%] Solve the homogeneous equation in (a). The answer contains symbol c.

(c) [20%] Assume $q(x)$ is continuous for all x. The problem $y' + y = q(x)$ can be solved using superposition $y = y_h + y_p$. The homogeneous equation $y' + y = 0$ has solution $y_h = ce^{-x}$. Give an equation for y_p, in terms of the function $q(x)$.

(a) $y' + py = q$ with $p = -\frac{1}{3x+1}$, $q = -16$

$\frac{(Wy)'}{W} = q$ with $W = e^{\int p\,dx} = e^{-\frac{1}{3}\ln|3x+1|}$

choose $W = (3x+1)^{-1/3}$ near $x = 0$.

$(Wy)' = qW$

$Wy = \int qW\,dx$ by quadrature

$= \int -16\,(3x+1)^{-1/3}dx$

$= -16\,\frac{(3x+1)^{2/3}}{2/3}\cdot\frac{1}{3} + c$

$y = c/W + (-8)(3x+1)$

$y(0) = -8$ gives $c = 0$. Then $\boxed{y = -8(3x+1)}$

(b) By (a), $y_h = \frac{c}{W}$. Then $\boxed{y_h = c\,(3x+1)^{1/3}}$

(c) $y' + y = q$

$\frac{(e^x y)'}{e^x} = q$

$(e^x y)' = q\,e^x$

$e^x y = \int q\,e^x\,dx$

$\boxed{y_p = e^{-x}\int_0^x q(r)e^r\,dr}$ indefinite integral is ok too.

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5. (Stability)

(a) [50%] Draw a phase line diagram for the differential equation

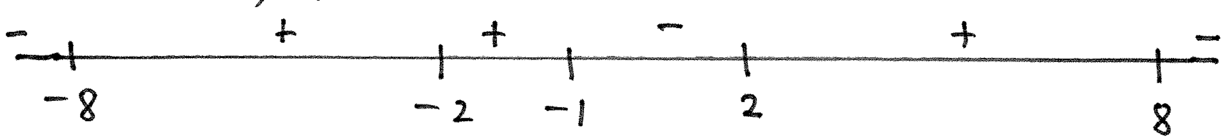
$$\frac{dx}{dt} = (2 + \sqrt[3]{x})\,(2 - \sqrt[3]{x})^3\,(x^2 + 3x + 2)(x^2 - 4)^3.$$

Expected in the diagram are equilibrium points and signs of x'.

(b) [50%] Draw a phase diagram using the phase line diagram of (a). Add these labels as appropriate: funnel, spout, node, source, sink, stable, unstable. Show at least 8 threaded curves. A direction field is not expected nor required.

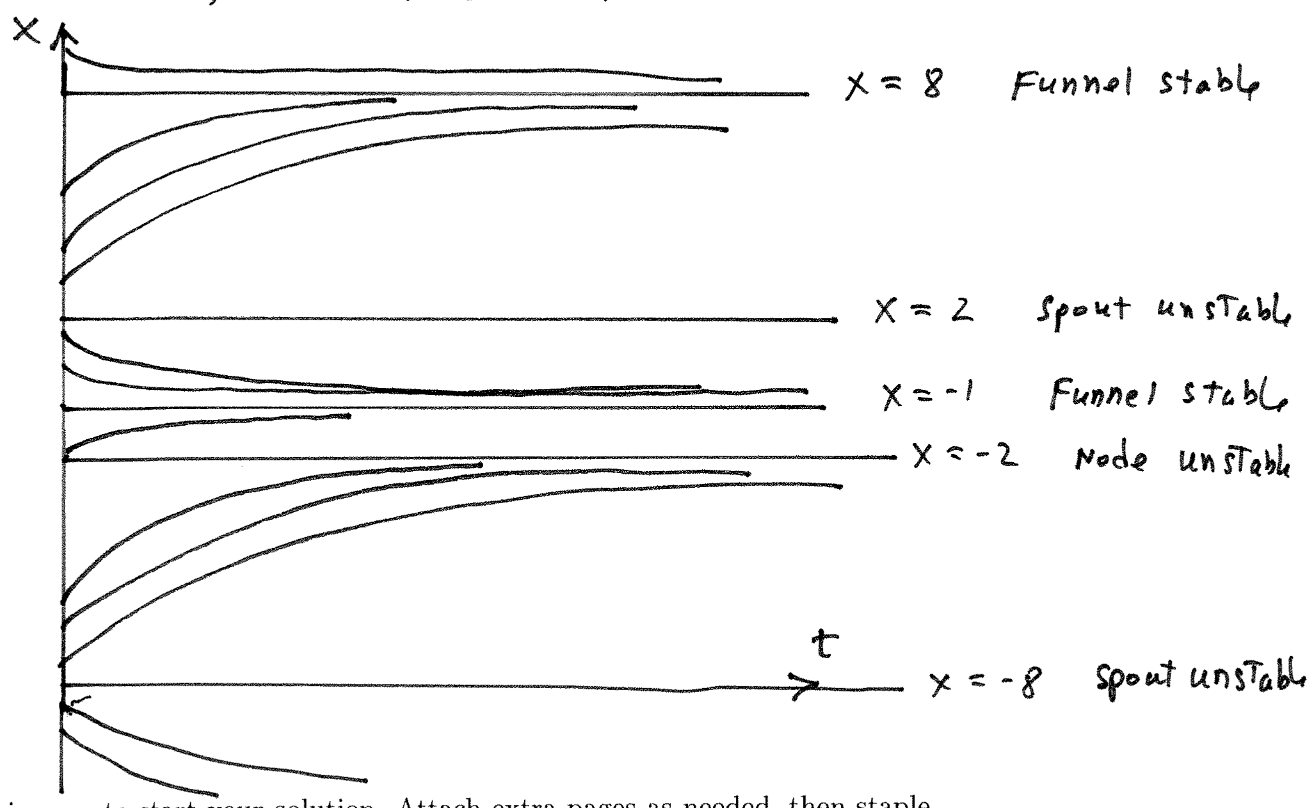
$$x' = (2 + x^{1/3})(2 - x^{1/3})^3 (x+2)^4 (x+1)(x-2)^3$$

roots : $-8, 8, -2, -1, 2$



with $f(x)$ standing for The RHS, Then

$f(-9) = -$, $f(-3) = +$, $f(-1.5) = +$, $f(0) = -$, $f(3) = +$, $f(9) = -$



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