Differential Equations and Linear Algebra 2250 [5:55pm]  
Midterm Exam 1  
Wednesday, 4 October 2006

Instructions: This in-class exam is 50 minutes. No calculators, notes, tables or books. No answer check is expected. Details count 75%. The answer counts 25%.

1. (Quadrature Equation)  
Solve for the general solution $y(x)$, $x > 0$, in the equation

$$y' = xe^{x^2} \sec^2 x - xe^{x^2} \tan^2 x + \sin x \cos x + \frac{x^3}{2 + x},$$

$$y' = xe^{x^2} (\sec^2 x - \tan^2 x) + \sin x \cos x + \frac{x^3}{2 + x},$$

$$y' = xe^{x^2} (1) + (\sin x) \cos x + x^2 - 2x + 4 + \frac{-8}{2 + x}$$

$$\int y' \, dx = \int xe^{x^2} \, dx + \int (\sin x) \cos x \, dx$$

$$+ \int (x^2 - 2x + 4) \, dx + \int -\frac{8 \, dx}{2 + x}$$

$$y = \frac{1}{2} e^{x^2} + \frac{\sin^2 x}{2} + \frac{x^3}{3} - x^2 + 4x - 8 \ln |2 + x| + C$$

Use this page to start your solution. Attach extra pages as needed, then staple.
2. (Classification of Equations)

The problem \( y' = f(x, y) \) is defined to be **separable** provided \( f(x, y) = F(x)G(y) \) for some functions \( F \) and \( G \).

(a) [60%] Check \( \square \) the problems that can be put into separable form, but don’t supply any details.

\[
\begin{array}{|c|c|}
\hline
\times \quad y' = -y(2y + x) + (x+1)y & e^y y' = xy^2 + x^2 \\
\times \quad y' = (\sin(x) e^{x+y})^2 & y' + \pi^2 y = e^{\pi x} \\
\hline
\end{array}
\]

(b) [15%] State a test which can verify that an equation \( y' = f(x, y) \) is linear but not quadrature.

(c) [25%] Use the separable equation test to show that \( y' = \sin(xy) \) is not separable.

\[\begin{align*}
(\text{b}) \quad \frac{df}{dy} & \text{ nonzero and independent of } y \\
& f(x, y) = q(x) - p(x)y \quad \text{with} \quad p(x) \neq 0
\end{align*}\]

\[\begin{align*}
(\text{c}) \quad \text{Choose } x_0 = 1, y_0 = \frac{\pi}{2}, \quad \text{Then } f(1, \frac{\pi}{2}) = \sin(\frac{\pi}{2}) = 1 \neq 0.
\end{align*}\]

Define

\[F = \frac{f(x, \frac{\pi}{2})}{f(1, \frac{\pi}{2})} = \sin(\frac{\pi x}{2})\]

\[G = f(1, y) = \sin(y)\]

Then

\[FG = \sin(\frac{\pi x}{2}) \sin(y)\]

\[\neq \sin(xy) = f\]

(because, otherwise, \( F(x, \frac{\pi}{2}) = 1 \) and \( G(y) = \sin(y) \) we get \( 1 = 1 \))

Therefore, \( y' = \sin(xy) \) is not separable.

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3. (Solve a Separable Equation)

Given \( y' = \left( \frac{\cos x}{\sin^2 x} + \frac{x}{1+x} \right) \tan^2 (1+y) \).

(a) Find all equilibrium solutions.
(b) Find the non-equilibrium solution in implicit form.

To save time, do not solve for \( y \) explicitly. Hint: change the tangent to sines and cosines and use \( \cos^2 \theta + \sin^2 \theta = 1 \).

(a) For \( y' = F(x) G(y) \), we solve \( G(c) = 0 \) for \( c \) and report all answers \( y = c, \) \( \tan \tan^2 (1+c) = 0, \) or \( 1+c = n\pi \) and: \( y = n\pi - 1 \) \( \text{for } n = 0, \pm 1, \pm 2, \ldots \)

(b) Start with \( \frac{y'}{G(y)} = F(x) \) and apply quadrature as follows.

\[
\int \frac{y'}{\tan^2 (1+y)} \, dx = \int \sin^{-1}(x) \cos(x) \, dx + \int (1 - \frac{1}{1+x}) \, dx
\]

\[
= -\frac{1}{\sin x} + x - \ln |1+x| + C
\]

\[
= -\csc x + x - \ln |1+x| + C
\]

Because \( \frac{1}{\tan^2 \theta} = \frac{\cos^2 \theta}{\sin^2 \theta} = \frac{1 - \sin^2 \theta}{\sin^2 \theta} = \csc^2 \theta - 1 \)

it follows that

\[
y - \cot(1+y) = x - \csc x - \ln |1+x| + C
\]

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4. (Linear Equations)

(a) [60%] Solve \( y' = -16 + \frac{1}{3x+1}y \), \( y(0) = -8 \). Show all integrating factor steps.

(b) [20%] Solve the homogeneous equation in (a). The answer contains symbol \( c \).

(c) [20%] Assume \( q(x) \) is continuous for all \( x \). The problem \( y' + y = q(x) \) can be solved using superposition \( y = y_h + y_p \). The homogeneous equation \( y' + y = 0 \) has solution \( y_h = ce^{-x} \). Give an equation for \( y_p \), in terms of the function \( q(x) \).

\[
\begin{align*}
(a) \quad y' + py &= q \\
\text{with} \quad p = \frac{-1}{3x+1}, \quad q = -16 \\
\frac{(wy)'}{w} &= q \\
\text{with} \quad w = e^{\int p \, dx} = e^{-\frac{1}{2} \ln(3x+1)} \\
\text{choose} \quad w = (3x+1)^{-1/3} \quad \text{near} \quad x = 0.
\end{align*}
\]

\[
\begin{align*}
(wy)' &= qw \\
w'y &= \int qw \, dx \quad \text{by quadrature} \\
&= \int -16 (3x+1)^{-1/3} \, dx \\
&= -16 \left(3x+1\right)^{2/3} \cdot \frac{1}{2} + c \\
g &= \frac{c}{w} + (-8)(3x+1) \\
y(0) &= -8 \quad \text{gives} \quad c = 0. \quad \text{Then} \\
&\underline{y_f = -8(3x+1)} \\
y_h &= c (3x+1)^{1/3} \\
&\underline{y_h = c (3x+1)^{1/3}}
\end{align*}
\]

(b) By (a), \( y_h = \frac{c}{w} \). Then

(c) \( y' + y = q \)

\[
\begin{align*}
\frac{(e^xy)'}{e^x} &= q \\
(e^xy)' &= qe^x \\
e^xy &= \int qe^x \, dx \\
y_p &= e^{-x} \int_0^x q(r)e^r \, dr \\
\text{indefinite integral is OK too.}
\end{align*}
\]

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5. (Stability)
(a) [50%] Draw a phase line diagram for the differential equation
\[
\frac{dx}{dt} = (2 + \sqrt[3]{x}) (2 - \sqrt[3]{x})^3 (x^2 + 3x + 2)(x^2 - 4)^3.
\]
Expected in the diagram are equilibrium points and signs of \(x'\).
(b) [50%] Draw a phase diagram using the phase line diagram of (a). Add these labels as appropriate: funnel, spout, node, source, sink, stable, unstable. Show at least 8 threaded curves. A direction field is not expected nor required.

\[x' = \frac{4}{(2 + x^{\sqrt{3}})(2 - x^{\sqrt{3}})^3 (x+2)(x+1)(x-2)^3}\]

Roots: \(-8, 8, -2, -1, 2\)

\[\begin{array}{cccccc}
-8 & + & -2 & -1 & 2 & + \\
- & + & - & + & - & +
\end{array}\]

With \(f(x)\) standing for the RHS, then
\[f(-8) = - , \ f(-2) = + , \ f(-1) = + , \ f(0) = - , \ f(2) = + , \ f(8) = -\]

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