Name. KEY　　　　　　　　　　Time of your class ________

Differential Equations and Linear Algebra 2250 [10:45]

Version 2

Midterm Exam 1

Tuesday, 3 October 2006

Instructions: This in-class exam is 50 minutes. No calculators, notes, tables or books. No answer check is expected. Details count 75%. The answer counts 25%.

1. **(Quadrature Equation)**

Solve for the general solution $y(x)$ in the equation $y' = \ln(1+x) + x\csc^2 x - x\cot^2 x + \frac{x^3}{4+x^2}$, for $x > 0$.

$F = F_1 + F_2 + F_3$　　$F_1 = \ln(1+x)$, $F_2 = x\csc^2 x - x\cot^2 x$, $F_3 = x^3/(4+x^2)$

$$\int F_1\,dx = \int \ln u\,du \qquad \begin{cases} u = 1+x \\ du = dx \end{cases}$$

$$= u\ln u - u + c_1$$

$$= (1+x)\ln(1+x) - x + c_2 \text{ for } x > 0$$

$$\int F_2\,dx = \int x\,dx$$

$$= \frac{x^2}{2} + c_3$$

because $\cos^2 x + \sin^2 x = 1$ divided by $\sin^2 x$ gives $\cot^2 x + 1 = \csc^2 x$

$$\int F_3\,dx = \int \left(x + \frac{-4x}{4+x^2}\right)dx$$

$$\begin{array}{r|l} & x \\ \hline 4+x^2 & x^3 \\ & x^3+4x \\ \hline & -4x \end{array}$$

$$= \frac{x^2}{2} + \int \frac{-2\,du}{u} \qquad \begin{cases} u = 4+x^2 \\ du = 2x\,dx \end{cases}$$

$$= \frac{x^2}{2} - 2\ln(4+x^2) + c_4$$

$$\int y'\,dx = \int F\,dx \qquad \text{method of quadrature}$$

$$y = \int F_1\,dx + \int F_2\,dx + \int F_3\,dx$$

$$y = (1+x)\ln(1+x) - x + x^2 - 2\ln(4+x^2) + c$$

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2. (Classification of Equations)
The problem $y' = f(x,y)$ is defined to be **separable** provided $f(x,y) = F(x)G(y)$ for some functions F and G.

(a) [60%] Check (☒) the problems that can be put into separable form, but don't supply any details.

☒ $y' = -y(2xy+1)+(x+1)y$	☒ $e^y y' = xy^2 + x$
☒ $y' = e^{x+2y}$	☒ $y' + \pi y = e^{\pi}$

(b) [15%] State a test which can verify that an equation $y' = f(x,y)$ is not linear and not quadrature.

(c) [25%] Use the separable equation test to show that $y' = e^{x+y} + 1$ is not separable.

(b) $\frac{\partial f}{\partial y}$ depends on y $\Rightarrow$ $y'=f(x,y)$ not linear, not quadr.

(c) Choose $x_0=y_0=0$ and let $f(x,y)=e^{x+y}+1$

Then $f(0,0)=2\neq 0$

Let $F(x)=\frac{f(x,0)}{f(0,0)}=\frac{e^x+1}{2}$

Let $G(y)=f(0,y)=e^y+1$

Then

$$\begin{aligned} FG &= \tfrac{1}{2}(e^x+1)(e^y+1) \\ &= \tfrac{1}{2}(e^{x+y}+e^y+e^x+1) \\ &\neq e^{x+y}+1 = f \end{aligned}$$

By the test, $y'=e^{x+y}+1$ is not separable.

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3. (Solve a Separable Equation)
Given $y' = \left(\frac{\sin x}{\cos^2 x} + \left(\frac{x+2}{1+x} \right)^2 \right) \sin y \tan y.$
(a) Find all equilibrium solutions.
(b) Find the non-equilibrium solution in implicit form.
To save time, **do not solve** for y explicitly.

(a) $\sin(c) \tan(c) = 0$ for $c = n\pi$, $n = 0, \pm 1, \pm 2, \ldots$
Equil sols are $\boxed{y = n\pi}$

(b) write as $y'/G(y) = F(x)$:

$$\csc y \cot y \, y' = \sin x (\cos x)^{-2} + \left(\frac{x+2}{x+1}\right)^2$$

Integrate, by method of Quadrature.

$$LHS = \int \csc y \cot y \, y' dx$$
$$= \int \csc u \cot u \, du \qquad \begin{cases} u = y(x) \\ du = y' dx \end{cases}$$
$$= -\csc u + c_1$$
$$RHS = \int \sin x (\cos x)^{-2} dx + \int \left(\frac{x+2}{x+1}\right)^2 dx$$
$$= \int w^{-2}(-dw) + \int \left(1 + \frac{1}{x+1}\right)^2 dx \qquad \begin{cases} w = \cos x \\ dw = -\sin x \, dx \end{cases}$$
$$= w^{-1} + \int \left[1 + \frac{2}{x+1} + (x+1)^{-2}\right] dx$$
$$= \sec x + x + 2 \ln / x+1 / - \frac{1}{x+1} + c_2$$

ans:

$$-\csc y = \sec x + x + 2 \ln / x+1 / - \frac{1}{x+1} + C$$

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4. (Linear Equations)

(a) [60%] Solve $x'(t) = -16 + \frac{1}{2t+5}x(t)$, $x(0) = -80$. Show all integrating factor steps.

(b) [10%] Solve the homogeneous equation $\frac{dy}{dx} = -2y$. The answer contains symbol c.

(c) [30%] The problem $2y' = -4y + f(x)$ can be solved using the answer y_h from (b) plus superposition $y = y_h + y_p$. Give an equation for y_p, in terms of the function $f(x)$.

(a) $x' + \frac{-1}{2t+5} x = -16$

$\frac{(Wx)'}{W} = -16$

$(Wx)' = -16W$

$\int (Wx)' dt = \int -16 W dt$

$Wx = -16 \int (2t+5)^{-1/2} dt$

$= -16 \frac{(2t+5)^{1/2}}{1/2} \cdot \frac{1}{2} + c$

$x = -16(2t+5) + c(2t+5)^{1/2}$

$-80 = -16(0+5) + c\sqrt{0+5}$

$\boxed{x = -16(2t+5)}$

$W = e^{\int \frac{-dt}{2t+5}}$

$= e^{-\frac{1}{2}\ln|2t+5|}$

$= e^{\ln|2t+5|^{-1/2}}$

$= (2t+5)^{-1/2}$ near $t=0$.

(b) $\boxed{y = c e^{-2x}}$ by the Growth-Decay recipe.

(c) $y' + 2y = \frac{1}{2} f(x)$

$\frac{(e^{2x}y)'}{e^{2x}} = \frac{1}{2} f(x)$

$(e^{2x}y)' = \frac{1}{2} e^{2x} f(x)$

$e^{2x} y = \int \frac{1}{2} e^{2x} f(x) dx$

$\boxed{y_p(x) = \frac{1}{2} e^{-2x} \int_{x_0}^{x} f(t)dt}$

for any x_0

Indefinite integral gets full credit

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5. (Stability)

(a) [50%] Draw a phase line diagram for the differential equation

$$\frac{dx}{dt} = (2 - \sqrt[3]{x})^2\,(1 + 2x)(1 - 4x^2)(16x^2 - 4)^3.$$

Expected in the diagram are equilibrium points and signs of x'.

(b) [50%] Draw a phase diagram using the phase line diagram of (a). Add these labels as appropriate: funnel, spout, node, source, sink, stable, unstable. Show at least 8 threaded curves. A direction field is not expected nor required.

(a) $f(x) = (2 - x^{1/3})^2 (1+2x)(1-2x)(1+2x)(64)(2x-1)^3(2x+1)^3$

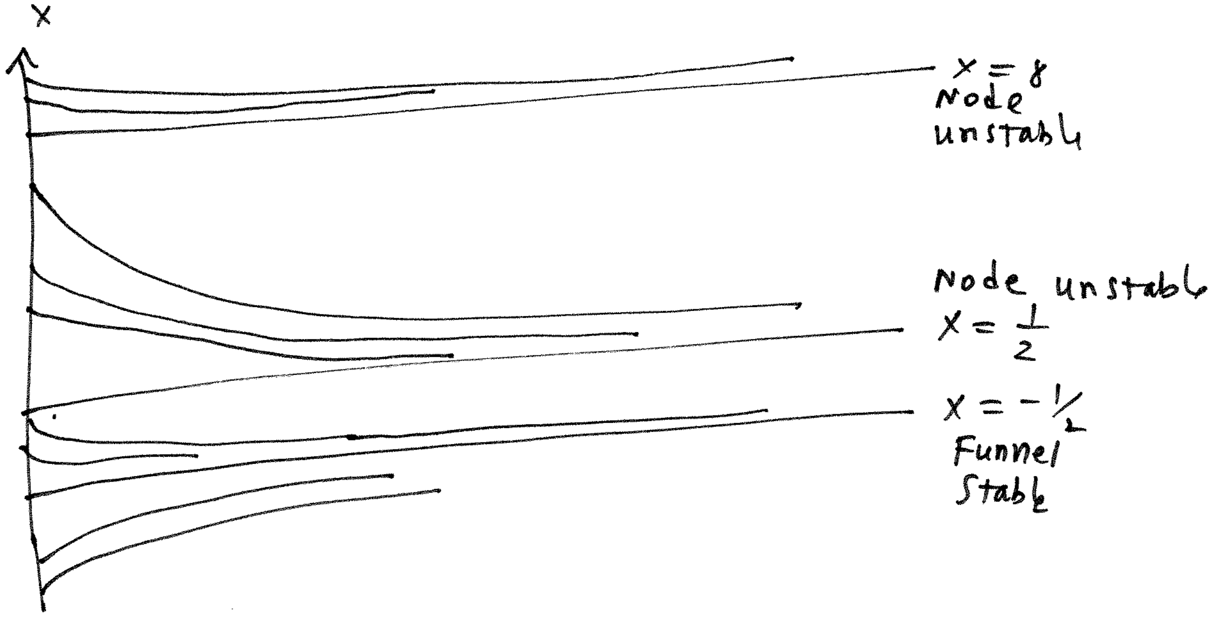
$f(x) = (2 - x^{1/3})^2 (1+2x)^5 (1-2x)^4 (-1)(64)$

roots $= 8, \frac{1}{2}, -\frac{1}{2}$

$f(9) = (+)(+)(+)(-) = -$, $f(1) = (+)(+1)(+)(-) = -$

$f(0) = (+)(+)(+)(-) = -$, $f(-1) = (+)(-)(+)(-) = +$

(b)



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