4.7 Cauchy's Kernel

The independent functions y_1 and y_2 in the general solution $y_h = c_1y_1 + c_2y_2$ of a homogeneous linear differential equation

$$a(x)y'' + b(x)y' + c(x)y = 0$$

are used to define **Cauchy's kernel**[1]

$$\mathcal{K}(x,t) = \frac{y_1(t)y_2(x) - y_1(x)y_2(t)}{y_1(t)y_2'(t) - y_1'(t)y_2(t)}. \tag{1}$$

The denominator is the *Wronskian* of y_1, y_2. Define

$$C_1(t) = \frac{-y_2(t)}{W(t)}, \quad C_2(t) = \frac{y_1(t)}{W(t)}. \tag{2}$$

Then Cauchy's kernel $\mathcal{K}$ has these properties (proved on page **??**):

$$\begin{array}{ll} \mathcal{K}(x,t) = C_1(t)y_1(x) + C_2(t)y_2(x), & \mathcal{K}(x,x) = 0, \\ \mathcal{K}_x(x,t) = C_1(t)y_1'(x) + C_2(t)y_2'(x), & \mathcal{K}_x(x,x) = 1, \\ \mathcal{K}_{xx}(x,t) = C_1(t)y_1''(x) + C_2(t)y_2''(x), & a\mathcal{K}_{xx} + b\mathcal{K}_x + c\mathcal{K} = 0. \end{array}$$

Theorem 15 (Cauchy Kernel Shortcut)
Let a, b, c be constants and let U be the unique solution of $aU''+bU'+cU = 0$, $U(0) = 0$, $U'(0) = 1$. Then Cauchy's kernel is $\mathcal{K}(x,t) = U(x-t)$.

The proof appears on page **??**.

Theorem 16 (Variation of Parameters Formula: Cauchy's Kernel)
Let a, b, c, f be continuous near $x = x_0$ and $a(x) \neq 0$. Let $\mathcal{K}$ be Cauchy's kernel for $ay'' + by' + cy = 0$. Then the non-homogeneous initial value problem

$$ay'' + by' + cy = f, \quad y(x_0) = y'(x_0) = 0$$

has solution

$$y_p(x) = \int_{x_0}^{x} \frac{\mathcal{K}(x,t)f(t)}{a(t)} dt.$$

The proof appears on page **??**. Specific initial conditions $y(x_0) = y'(x_0) = 0$ imply that y_p can be determined in a laboratory with just one experimental setup. The integral form of y_p shows that it depends *linearly* on the input $f(x)$.

22 **Example (Cauchy Kernel)** Verify that the equation $2y'' - y' - y = 0$ has Cauchy kernel $\mathcal{K}(x,t) = \frac{2}{3}(e^{x-t} - e^{-(x-t)/2})$.

[1] Pronunciation *ko–she*.

Solution: The two independent solutions y_1, y_2 are calculated from the *recipe*, which uses the characteristic equation $2r^2 - r - 1 = 0$. The roots are $-1/2$ and 1. The general solution is $y = c_1 e^{-x/2} + c_2 e^x$. Therefore, $y_1 = e^{-x/2}$ and $y_2 = e^x$.

The Cauchy kernel is the quotient

$$\begin{aligned}
\mathcal{K}(x,t) &= \frac{y_1(t)y_2(x) - y_1(x)y_2(t)}{y_1(t)y_2'(t) - y_1'(t)y_2(t)} && \text{Definition page ??.}\\
&= \frac{e^{-t/2}e^x - e^{-x/2}e^t}{e^{-t/2}e^t + 0.5e^{-t/2}e^t} && \text{Substitute } y_1 = e^{-x/2},\ y_2 = e^x.\\
&= \frac{2}{3}(e^{-t}e^x - e^{-x/2}e^{t/2}) && \text{Simplify.}\\
&= \frac{2}{3}(e^{x-t} - e^{(t-x)/2}) && \text{Final answer.}
\end{aligned}$$

An alternative method to determine the Cauchy kernel is to apply the shortcut Theorem **??**. We will apply it to check the answer. Solution U must be $U(x) = Ay_1(x) + By_2(x)$ for some constants A, B, determined by the conditions $U(0) = 0$, $U'(0) = 1$. The resulting equations for A, B are $A + B = 0$, $-A/2 + B = 1$. Solving gives $-A = B = 2/3$ and then $U(x) = \frac{2}{3}(e^x - e^{-x/2})$. The kernel is $\mathcal{K}(x,t) = U(x-t) = \frac{2}{3}(e^{x-t} - e^{-(x-t)/2})$.

23 Example (Variation of Parameters) Solve $y'' = |x|$ by Cauchy kernel methods, verifying $y = c_1 + c_2 x + |x|^3/6$.

Solution: First, an independent method will be described, in order to provide a check on the solution. The method involves splitting the equation into two problems $y'' = x$ and $y'' = -x$. The two polynomial answers $y = x^3/6$ on $x > 0$ and $y = -x^3/6$ on $x < 0$, obtained by quadrature, are re-assembled to obtain a single formula $y_p(x) = |x|^3/6$ valid on $-\infty < x < \infty$.

The Cauchy kernel method will be applied to verify the general solution $y = c_1 + c_2 x + |x|^3/6$.

Homogeneous solution. The *recipe* for constant equations, applied to $y'' = 0$, gives $y_h = c_1 + c_2 x$. Suitable independent solutions are $y_1(x) = 1$, $y_2(x) = x$.

Cauchy kernel for $y'' = 0$. It is computed by formula, $\mathcal{K}(x,t) = ((1)(x) - (t)(1))/(1)$ or $\mathcal{K}(x,t) = x - t$.

Variation of parameters. The solution is $y_p(x) = |x|^3/6$, by Theorem **??**, details below.

$$\begin{aligned}
y_p(x) &= \textstyle\int_0^x \mathcal{K}(x,t)|t|dt && \text{Theorem ??, page ??.}\\
&= \textstyle\int_0^x (x-t)t\,dt && \text{Substitute } \mathcal{K} = x - t \text{ and } |t| = t \text{ for } x > 0.\\
&= x\textstyle\int_0^x t\,dt - \int_0^x t^2 dt && \text{Split into two integrals.}\\
&= x^3/6 && \text{Evaluate for } x > 0.
\end{aligned}$$

If $x < 0$, then the evaluation differs only by $|t| = -t$ in the integrand. This gives $y_p(x) = -x^3/6$ for $x < 0$. The two formulas can be combined into $y_p(x) = |x|^3/6$, valid for $-\infty < x < \infty$.

24 Example (Two Methods) Solve $y'' - y = e^x$ by undetermined coefficients and by variation of parameters. Explain any differences in the answers.

Solution:

Homogeneous solution. The characteristic equation $r^2 - 1 = 0$ for $y'' - y = 0$ has roots ± 1. The homogeneous solution is $y_h = c_1 e^x + c_2 e^{-x}$.

Undetermined Coefficients Summary. The general solution is reported to be $y = y_h + y_p = c_1 e^x + c_2 e^{-x} + xe^x/2$.

Kümmer's polynomial $\times$ exponential method applies to give $y = e^x Y$ and $[(D+1)^2 - 1]Y = 1$. The latter simplifies to $Y'' + 2Y' = 1$, which has polynomial solution $Y = x/2$. Then $y_p = xe^x/2$.

Variation of Parameters Summary. The homogeneous solution $y_h = c_1 e^x + c_2 e^{-x}$ found above implies $y_1 = e^x$, $y_2 = e^{-x}$ is a suitable independent pair of solutions, because their Wronskian is $W = -2$

The Cauchy kernel is given by $\mathcal{K}(x,t) = \frac{1}{2}(e^{x-t} - e^{t-x})$, details below. The shortcut Theorem **??** also applies with $U(x) = \sinh(x) = (e^x - e^{-x})/2$. The variation of parameters formula is applied from Theorem **??**: $y_p(x) = \int_0^x \mathcal{K}(x,t)e^t dt$. It evaluates to $y_p(x) = xe^x/2 - (e^x - e^{-x})/4$, details below.

Differences. The two methods give respectively $y_p = xe^x/2$, and $y_p = xe^x/2 - (e^x - e^{-x})/4$. The solutions $y_p = xe^x/2$ and $y_p = xe^x/2 - (e^x - e^{-x})/4$ differ by the homogeneous solution $(e^x - e^{-x})/4$. In both cases, the general solution is

$$y = c_1 e^x + c_2 e^{-x} + \frac{1}{2}xe^x,$$

because terms of the homogeneous solution can be absorbed into the arbitrary constants c_1, c_2.

Computational Details.

$$\begin{aligned}
\mathcal{K}(x,t) &= \frac{y_1(t)y_2(x) - y_1(x)y_2(t)}{y_1(t)y_2'(t) - y_1'(t)y_2(t)} && \text{Definition page ??.} \\
&= \frac{e^t e^{-x} - e^x e^{-t}}{e^t(-e^{-t}) - e^t e^{-t}} && \text{Substitute.} \\
&= \frac{1}{2}(e^{x-t} - e^{t-x}) && \text{Cauchy kernel found.}
\end{aligned}$$

$$\begin{aligned}
y_p(x) &= \int_0^x \mathcal{K}(x,t)e^t dt && \text{Theorem ??, page ??.} \\
&= \frac{1}{2}\int_0^x (e^{x-t} - e^{t-x})e^t dt && \text{Substitute } \mathcal{K} = \tfrac{1}{2}(e^{x-t} - e^{t-x}). \\
&= \frac{1}{2}e^x \int_0^x dt - \frac{1}{2}\int_0^x e^{2t-x} dt && \text{Split into two integrals.} \\
&= \frac{1}{2}xe^x - \frac{1}{4}(e^x - e^{-x}) && \text{Evaluation completed.}
\end{aligned}$$

Proofs of Cauchy Kernel Properties. The equation $\mathcal{K}(x,t) = C_1(t)y_1(x) + C_2(t)y_2(x)$ is an algebraic identity, using the definitions of C_1 and C_2. Then $\mathcal{K}(x,x)$ is a fraction with numerator $y_1(x)y_2(x) - y_1(x)y_2(x) = 0$, giving the second identity $\mathcal{K}(x,x) = 0$.

The partial derivative formula $\mathcal{K}_x(x,t) = C_1(t)y_1'(x) + C_2(t)y_2'(x)$ is obtained by ordinary differentiation on x in the previous identity. Then $\mathcal{K}_x(x,x)$ is a fraction with numerator $y_1(x)y_2'(x) - y_1'(x)y_2(x)$, which exactly cancels the denominator, giving the identity $\mathcal{K}_x(x,x) = 1$.

The second derivative formula $\mathcal{K}_{xx}(x,t) = C_1(t)y_1''(x) + C_2(t)y_2''(x)$ results by ordinary differentiation on x in the formula for $\mathcal{K}_x$. The differential equation $a\mathcal{K}_{xx} + b\mathcal{K}_x + c\mathcal{K} = 0$ is satisfied, because $\mathcal{K}$ in the variable x is a linear combination of y_1 and y_2, which are given to be solutions.

Proof of Theorem ??: Let $y(x) = \mathcal{K}(x,t) - U(x-t)$ for fixed t. It will be shown that y is a solution and $y(t) = y'(t) = 0$. Already known from page **??** is the relation $a\mathcal{K}_{xx}(x,t) + b\mathcal{K}_x(x,t) + c\mathcal{K}(x,t) = 0$. By assumption, $aU''(x-t)+bU'(x-t)+cU(x-t) = 0$. By the chain rule, both terms in y satisfy the differential equation, hence y is a solution. At $x = t$, $y(t) = \mathcal{K}(t,t)-U(0) = 0$ and $y'(t) = K_x(t,t) - U'(0) = 0$ (see page **??**). Then y is a solution of the homogeneous equation with zero initial conditions. By uniqueness, $y(x) \equiv 0$, which proves $\mathcal{K}(x,t) = U(x-t)$.

Proof of Theorem ??: Let $F(t) = f(t)/a(t)$. It will be shown that y_p as given has two continuous derivatives given by the integral formulas

$$y_p'(x) = \int_{x_0}^{x} \mathcal{K}_x(x,t)F(t)dt, \quad y_p''(x) = \int_{x_0}^{x} \mathcal{K}_{xx}(x,t)F(t)dt + F(x).$$

Then

$$ay_p'' + by_p' + cy_p = \int_{x_0}^{x} (a\mathcal{K}_{xx} + b\mathcal{K}_x + c\mathcal{K})F(t)dt + aF.$$

The equation $a\mathcal{K}_{xx} + b\mathcal{K}_x + c\mathcal{K} = 0$, page **??**, shows the integrand on the right is zero. Therefore $ay_p'' + by_p' + cy_p = f(x)$, which would complete the proof.

Needed for the calculation of the derivative formulas is the fundamental theorem of calculus relation $\left(\int_{x_0}^{x} G(t)dt\right)' = G(x)$, valid for continuous G. The product rule from calculus can be applied directly, because y_p is a sum of products:

$$\begin{aligned}
y_p' &= \left(y_1(x)\int_{x_0}^{x} C_1F dt + y_2(x)\int_{x_0}^{x} C_2F dt\right)' \\
&= y_1'\int_{x_0}^{x} C_1F dt + y_2'\int_{x_0}^{x} C_2F dt + y_1(x)C_1(x)F(x) + y_2(x)C_2(x)F(x) \\
&= y_1'\int_{x_0}^{x} C_1F dt + y_2'\int_{x_0}^{x} C_2F dt + \mathcal{K}(x,x)F(x) \\
&= \int_{x_0}^{x} \mathcal{K}_x(x,t)F(t)dt
\end{aligned}$$

The terms contributed by differentiation of the integrals add to zero because $\mathcal{K}(x,x) = 0$ (page **??**).

$$\begin{aligned}
y_p'' &= \left(y_1'(x)\int_{x_0}^{x} C_1F dt + y_2'(x)\int_{x_0}^{x} C_2F dt\right)' \\
&= y_1''\int_{x_0}^{x} C_1F dt + y_2''\int_{x_0}^{x} C_2F dt + y_1'(x)C_1(x)F(x) + y_2'(x)C_2(x)F(x) \\
&= y_1''\int_{x_0}^{x} C_1F dt + y_2''\int_{x_0}^{x} C_2F dt + \mathcal{K}_x(x,x)F(x) \\
&= \int_{x_0}^{x} \mathcal{K}_{xx}(x,t)F(t)dt + F(x)
\end{aligned}$$

The terms contributed by differentiation of the integrals add to $F(x)$ because $\mathcal{K}_x(x,x) = 1$ (page **??**).

Exercises 4.7

Cauchy Kernel. Find the Cauchy kernel $\mathcal{K}(x,t)$ for the given homogeneous differential equation.

23. $y'' - y = 0$

24. $y'' - 4y = 0$

25. $y'' + y = 0$

26. $y'' + 4y = 0$

27. $4y'' + y' = 0$

28. $y'' + y' = 0$

29. $y'' + y' + y = 0$

30. $y'' - y' + y = 0$

Variation of Parameters. Find the general solution $y_h + y_p$ by applying a variation of parameters formula.

35. $y'' = x^2$

36. $y'' = x^3$

37. $y'' + y = \sin x$

38. $y'' + y = \cos x$

39. $y'' + y' = \ln|x|$

40. $y'' + y' = -\ln|x|$

41. $y'' + 2y' + y = e^{-x}$

42. $y'' - 2y' + y = e^{x}$