**Three Examples.** It is possible to solve a variety of differential equations without reading a differential equations textbook:

Growth-Decay $$\frac{dA}{dt} = kA(t),\ A(0) = A_0$$

$$\boxed{A(t) = A_0 e^{kt}}$$

Newton Cooling $$\frac{du}{dt} = -h(u(t) - u_1),\ u(0) = u_0$$

$$\boxed{u(t) = u_1 + (u_0 - u_1)e^{-ht}}$$

Verhulst Logistic $$\frac{dP}{dt} = (a - bP(t))P(t),\ P(0) = P_0$$

$$\boxed{P(t) = \frac{aP_0}{bP_0 + (a - bP_0)e^{-at}}}$$

Like the multiplication tables in elementary school, these models and their solution formulas should be *memorized*, in order to form a foundation of intuition for all differential equation theory. The last two solutions are derived from the growth-decay equation by variable changes: $A(t) = u(t) - u_1$ and $A(t) = P(t)/(a - bP(t))$.

## Fundamental Theorem of Calculus

(a) $\int_a^b f'(x)dx = f(b) - f(a)$

(b) $\left(\int_a^x g(t)dt\right)' = g(x)$

Isaac Newton found these formulas in an effort to extend the formula D=RT to the case of instantaneous rates.

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## The method of Quadrature

- Applies to equations like $y' = 2x$
- Uses the fundamental theorem of calculus
- only produces a candidate solution — it does not verify the solution.

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**Example** Solve by the method of Quadrature

$$y'' = 2x$$

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Solution:

$\int y'\,dx = \int 2x\,dx$ Integrate both sides on x.

$y(x) + C_1 = x^2 + C_2$ Apply fund. Thm. Calc.

$y(x) = x^2 + C$ Collect constants. Candidate found.

**1 Example (Decay Law Derivation)** Derive the decay law $\frac{dA}{dt} = kA(t)$ from the sentence

*Radioactive material decays at a rate proportional to the amount present.*

**Solution**: The sentence is first dissected into English phrases **1** to **4**.

| | |
|---|---|
| **1**: *Radioactive material* | The phrase causes the invention of a symbol $A$ for the amount present at time $t$. |
| **2**: *decays at a rate* | It means $A$ undergoes decay. Then $A$ changes. Calculus conventions imply the rate of change is $dA/dt$. |
| **3**: *proportional to* | Literally, it means *equal to a constant multiple of.* Let $k$ be the proportionality constant. |
| **4**: *the amount present* | The amount of radioactive material present is $A(t)$. |

**Solution**: *Continued . . .*

The four phrases are translated into mathematical notation as follows.

| | |
|---|---|
| **Phrases 1 and 2** | Symbol $dA/dt$. |
| **Phrase 3** | Equal sign '$=$' and a constant $k$. |
| **Phrase 4** | Symbol $A(t)$. |

Let $A(t)$ be the amount present at time $t$. The translation is $\dfrac{dA}{dt} = kA(t)$.

# Background

| | |
|---|---|
| $\ln e^x = x,\ \ e^{\ln y} = y$ | In words, the exponential and the logarithm are inverses. The domains are $-\infty < x < \infty$, $0 < y < \infty$. |
| $e^0 = 1,\ \ \ln(1) = 0$ | Special values, usually memorized. |
| $e^{a+b} = e^a e^b$ | In words, the exponential of a sum of terms is the product of the exponentials of the terms. |
| $(e^a)^b = e^{ab}$ | Negatives are allowed, e.g., $(e^a)^{-1} = e^{-a}$. |
| $\left(e^{u(t)}\right)' = u'(t)e^{u(t)}$ | The *chain rule* of calculus implies this formula from the identity $(e^x)' = e^x$. |
| $\ln AB = \ln A + \ln B$ | In words, the logarithm of a product of factors is the sum of the logarithms of the factors. |
| $B\ln(A) = \ln\left(A^B\right)$ | Negatives are allowed, e.g., $-\ln A = \ln \frac{1}{A}$. |
| $(\ln\|u(t)\|)' = \frac{u'(t)}{u(t)}$ | The identity $(\ln(x))' = 1/x$ implies this general version by the *chain rule*. |