5.4 Variation of Parameters

The **method of variation of parameters** applies to solve

$$a(x)y'' + b(x)y' + c(x)y = f(x). \tag{1}$$

Continuity of a, b, c and f is assumed, plus $a(x) \neq 0$. The method is important because it solves the largest class of equations. Specifically *included* are functions $f(x)$ like $\ln|x|$, $|x|$, e^{x^2}.

Homogeneous Equation. The method of variation of parameters uses facts about the homogeneous differential equation

$$a(x)y'' + b(x)y' + c(x)y = 0. \tag{2}$$

The success depends upon writing the general solution of (2) as

$$y = c_1 y_1(x) + c_2 y_2(x) \tag{3}$$

where y_1, y_2 are *known functions* and c_1, c_2 are arbitrary constants. If a, b, c are constants, then the standard *recipe* for (2) finds y_1, y_2. It is known that y_1, y_2 as reported by the recipe are *independent*.

Independence. Two solutions y_1, y_2 of (2) are called **independent** if neither is a constant multiple of the other. The term **dependent** means *not independent*, in which case either $y_1(x) = cy_2(x)$ or $y_2(x) = cy_1(x)$ holds for all x, for some constant c. Independence can be tested through the **Wronskian** of y_1, y_2, defined by

$$W(x) = y_1(x)y_2'(x) - y_1'(x)y_2(x).$$

Theorem 10 (Wronskian and Independence)
The Wronskian of two solutions satisfies $a(x)W' + b(x)W = 0$, which implies **Abel's identity**

$$W(x) = W(x_0)e^{-\int_{x_0}^{x}(b(t)/a(t))dt}.$$

Two solutions of (2) are independent if and only if $W(x) \neq 0$.

The proof appears on page 205.

Theorem 11 (Variation of Parameters Formula)
Let a, b, c, f be continuous near $x = x_0$ and $a(x) \neq 0$. Let y_1, y_2 be two independent solutions of the homogeneous equation $ay'' + by' + cy = 0$ and let $W(x) = y_1(x)y_2'(x) - y_1'(x)y_2(x)$. Then the non-homogeneous differential equation

$$ay'' + by' + cy = f$$

has a particular solution

$$y_p(x) = y_1(x)\left(\int \frac{y_2(x)(-f(x))}{a(x)W(x)}dx\right) + y_2(x)\left(\int \frac{y_1(x)f(x)}{a(x)W(x)}dx\right). \tag{4}$$

The proof is delayed to page 205.

History of Variation of Parameters. The solution y_p was discovered by varying the constants c_1, c_2 in the homogeneous solution (3), assuming they depend on x. This results in formulas $c_1(x) = \int C_1 F$, $c_2(x) = \int C_2 F$ where $F(x) = f(x)/a(x)$, $C_1(t) = \dfrac{-y_2(t)}{W(t)}$,$C_2(t) = \dfrac{y_1(t)}{W(t)}$; see the historical details on page 205. Then

$$
\begin{aligned}
y &= y_1(x)\int C_1 F + y_2(x)\int C_2 F && \text{Substitute in (3) for } c_1,\ c_2.\\
&= -y_1(x)\int y_2\frac{F}{W} + y_2(x)\int y_1\frac{F}{W} && \text{Use (2) for } C_1,\ C_2.\\
&= \int (y_2(x)y_1(t) - y_1(x)y_2(t))\frac{F(t)}{W(t)}dt && \text{Collect on } F/W.\\
&= \int \frac{y_1(t)y_2(x) - y_1(x)y_2(t)}{y_1(t)y_2'(t) - y_1'(t)y_2(t)}F(t)dt && \text{Expand } W = y_1y_2' - y_1'y_2.
\end{aligned}
$$

Any one of the last three equivalent formulas is called a **classical variation of parameters formula**. The fraction in the last integrand is called Cauchy's kernel. We prefer the first, equivalent to equation (4), for ease of use.

5 Example (Independence) Consider $y'' - y = 0$. Show the two solutions $\sinh(x)$ and $\cosh(x)$ are independent using Wronskians.

Solution: Let $W(x)$ be the Wronskian of $\sinh(x)$ and $\cosh(x)$. The calculation below shows $W(x) = -1$. By Theorem 10, the solutions are independent.

Background. The calculus *definitions* for hyperbolic functions are $\sinh x = (e^x - e^{-x})/2$, $\cosh x = (e^x + e^{-x})/2$. Their derivatives are $(\sinh x)' = \cosh x$ and $(\cosh x)' = \sinh x$. For instance, $(\cosh x)'$ stands for $\frac{1}{2}(e^x + e^{-x})'$, which evaluates to $\frac{1}{2}(e^x - e^{-x})$, or $\sinh x$.

Wronskian detail. Let $y_1 = \sinh x$, $y_2 = \cosh x$. Then

$$
\begin{aligned}
W &= y_1(x)y_2'(x) - y_1'(x)y_2(x) && \text{Definition of Wronskian } W.\\
&= \sinh(x)\sinh(x) - \cosh(x)\cosh(x) && \text{Substitute for } y_1,\ y_1',\ y_2,\ y_2'.\\
&= \tfrac{1}{4}(e^x - e^{-x})^2 - \tfrac{1}{4}(e^x + e^{-x})^2 && \text{Apply exponential definitions.}\\
&= -1 && \text{Expand and cancel terms.}
\end{aligned}
$$

6 Example (Wronskian) Given $2y'' - xy' + 3y = 0$, verify that a solution pair y_1, y_2 has Wronskian $W(x) = W(0)e^{x^2/4}$.

Solution: Let $a(x) = 2$, $b(x) = -x$, $c(x) = 3$. The Wronskian is a solution of $W' = -(b/a)W$, hence $W' = xW/2$. The solution is $W = W(0)e^{x^2/4}$, by growth-decay theory.

7 Example (Variation of Parameters) Solve $y'' + y = \sec x$ by variation of parameters, verifying $y = c_1 \cos x + c_2 \sin x + x \sin x + \cos(x) \ln|\cos x|$.

Solution:
Homogeneous solution y_h. The *recipe* for constant equation $y'' + y = 0$ is applied. The characteristic equation $r^2 + 1 = 0$ has roots $r = \pm i$ and $y_h = c_1 \cos x + c_2 \sin x$.

Wronskian. Suitable independent solutions are $y_1 = \cos x$ and $y_2 = \sin x$, taken from the *recipe*. Then $W(x) = \cos^2 x + \sin^2 x = 1$.

Calculate y_p. The variation of parameters formula (4) is applied. The integration proceeds near $x = 0$, because $\sec(x)$ is continuous near $x = 0$.

$$\begin{aligned} y_p(x) &= -y_1(x) \int y_2(x) \sec(x) dx + y_2(x) \int y_1(x) \sec x dx && \boxed{1} \\ &= -\cos x \int \tan(x) dx + \sin x \int 1 dx && \boxed{2} \\ &= x \sin x + \cos(x) \ln|\cos x| && \boxed{3} \end{aligned}$$

Details: $\boxed{1}$ Use equation (4). $\boxed{2}$ Substitute $y_1 = \cos x$, $y_2 = \sin x$. $\boxed{3}$ Integral tables applied. Integration constants set to zero.

8 Example (Two Methods) Solve $y'' - y = e^x$ by undetermined coefficients and by variation of parameters. Explain any differences in the answers.

Solution: The general solution is reported to be $y = y_h + y_p = c_1 e^x + c_2 e^{-x} + xe^x/2$. Details follow.

Homogeneous solution. The characteristic equation $r^2 - 1 = 0$ for $y'' - y = 0$ has roots ± 1. The homogeneous solution is $y_h = c_1 e^x + c_2 e^{-x}$.

Undetermined Coefficients Summary. The basic trial solution method gives initial trial solution $y = d_1 e^x$, because the RHS $= e^x$ has all derivatives given by a linear combination of the independent function e^x. The fixup rule applies because the homogeneous solution contains duplicate term $c_1 e^x$. The final trial solution is $y = d_1 x e^x$. Substitution into $y'' - y = e^x$ gives $2d_1 e^x + d_1 x e^x - d_1 x e^x = e^x$. Cancel e^x and equate coefficients of powers of x to find $d_1 = 1/2$. Then $y_p = xe^x/2$.

Variation of Parameters Summary. The homogeneous solution $y_h = c_1 e^x + c_2 e^{-x}$ found above implies $y_1 = e^x$, $y_2 = e^{-x}$ is a suitable independent pair of solutions. Their Wronskian is $W = -2$

The variation of parameters formula (11) applies:

$$y_p(x) = e^x \int \frac{-e^{-x}}{-2} e^x dx + e^{-x} \int \frac{e^x}{-2} e^x dx.$$

Integration, followed by setting all constants of integration to zero, gives $y_p(x) = xe^x/2 - e^x/4$.

Differences. The two methods give respectively $y_p = xe^x/2$ and $y_p(x) = xe^x/2 - e^x/4$. The solutions $y_p = xe^x/2$ and $y_p(x) = xe^x/2 - e^x/4$ differ by the homogeneous solution $-xe^x/4$. In both cases, the general solution is

$$y = c_1 e^x + c_2 e^{-x} + \frac{1}{2} x e^x,$$

because terms of the homogeneous solution can be absorbed into the arbitrary constants c_1, c_2.

Proof of Theorem 10: The function $W(t)$ given by Abel's identity is the unique solution of the growth-decay equation $W' = -(b(x)/a(x))W$; see page 3. It suffices then to show that W satisfies this differential equation. The details:

$W' = (y_1y_2' - y_1'y_2)'$	Definition of Wronskian.
$= y_1y_2'' + y_1'y_2' - y_1''y_2 - y_1'y_2'$	Product rule; $y_1'y_2'$ cancels.
$= y_1(-by_2' - cy_2)/a - (-by_1' - cy_1)y_2/a$	Both y_1, y_2 satisfy (2).
$= -b(y_1y_2' - y_1'y_2)/a$	Cancel common cy_1y_2/a.
$= -bW/a$	Verification completed.

The independence statement will be proved from the contrapositive: $W(x) = 0$ for all x if and only if y_1, y_2 are not independent. Technically, independence is defined relative to the common domain of the graphs of y_1, y_2 and W. Henceforth, *for all* x means for all x in the common domain.

Let y_1, y_2 be two solutions of (2), not independent. By re-labelling as necessary, $y_1(x) = cy_2(x)$ holds for all x, for some constant c. Differentiation implies $y_1'(x) = cy_2'(x)$. Then the terms in $W(x)$ cancel, giving $W(x) = 0$ for all x.

Conversely, let $W(x) = 0$ for all x. If $y_1 \equiv 0$, then $y_1(x) = cy_2(x)$ holds for $c = 0$ and y_1, y_2 are not independent. Otherwise, $y_1(x_0) \neq 0$ for some x_0. Define $c = y_2(x_0)/y_1(x_0)$. Then $W(x_0) = 0$ implies $y_2'(x_0) = cy_1'(x_0)$. Define $y = y_2 - cy_1$. By linearity, y is a solution of (2). Further, $y(x_0) = y'(x_0) = 0$. By uniqueness of initial value problems, $y \equiv 0$, that is, $y_2(x) = cy_1(x)$ for all x, showing y_1, y_2 are not independent.

Proof of Theorem 11: Let $F(t) = f(t)/a(t)$, $C_1(x) = -y_2(x)/W(x)$, $C_2(x) = y_1(x)/W(x)$. Then y_p as given in (4) can be differentiated twice using the product rule and the fundamental theorem of calculus rule $(\int g)' = g$. Because $y_1C_1 + y_2C_2 = 0$ and $y_1'C_1 + y_2'C_2 = 1$, then y_p and its derivatives are given by

$$\begin{aligned} y_p(x) &= y_1 \textstyle\int C_1F dx + y_2 \int C_2F dx, \\ y_p'(x) &= y_1' \textstyle\int C_1F dx + y_2' \int C_2F dx, \\ y_p''(x) &= y_1'' \textstyle\int C_1F dx + y_2'' \int C_2F dx + F(x). \end{aligned}$$

Let $F_1 = ay_1'' + by_1' + cy_1$, $F_2 = ay_2'' + by_2' + cy_2$. Then

$$ay_p'' + by_p' + cy_p = F_1 \int C_1F dx + F_2 \int C_2F dx + aF.$$

Because y_1, y_2 are solutions of the homogeneous differential equation, then $F_1 = F_2 = 0$. By definition, $aF = f$. Therefore,

$$ay_p'' + by_p' + cy_p = f.$$

The proof is complete.

Historical Details. The original variation ideas, attributed to Joseph Louis Lagrange (1736-1813), involve substitution of $y = c_1(x)y_1(x) + c_2(x)y_2(x)$ into (1) plus imposing an extra condition on the unknowns c_1, c_2:

$$c_1'y_1 + c_2'y_2 = 0.$$

The product rule gives $y' = c_1'y_1 + c_1y_1' + c_2'y_2 + c_2y_2'$, which then reduces to the two-termed expression $y' = c_1y_1' + c_2y_2'$. Substitution into (1) gives

$$a(c_1'y_1' + c_1y_1'' + c_2'y_2' + c_2y_2'') + b(c_1y_1' + c_2y_2') + c(c_1y_1 + c_2y_2) = f$$

which upon collection of terms becomes

$$c_1(ay_1'' + by_1' + cy_1) + c_2(ay_2'' + by_2' + cy_2) + ay_1'c_1' + ay_2'c_2' = f.$$

The first two groups of terms vanish because y_1, y_2 are solutions of the homogeneous equation, leaving just $ay_1'c_1' + ay_2'c_2' = f$. There are now two equations and two unknowns $X = c_1'$, $Y = c_2'$:

$$\begin{array}{rcrcl} ay_1'X & + & ay_2'Y & = & f, \\ y_1X & + & y_2Y & = & 0. \end{array}$$

Solving by elimination,

$$X = \frac{-y_2 f}{aW}, \quad Y = \frac{y_1 f}{aW}.$$

Then c_1 is the integral of X and c_2 is the integral of Y, which completes the historical account of the relations

$$c_1(x) = \int \frac{-y2(x)f(x)}{a(x)W(x)}dx, \quad c_2(x) = \int \frac{y_1(x)f(x)}{a(x)W(x)}dx.$$

Exercises 5.4

Independence. Find solutions y_1, y_2 of the given homogeneous differential equation which are independent by the Wronskian test, page 202.

1. $y'' - y = 0$

2. $y'' - 4y = 0$

3. $y'' + y = 0$

4. $y'' + 4y = 0$

5. $4y'' = 0$

6. $y'' = 0$

7. $4y'' + y' = 0$

8. $y'' + y' = 0$

9. $y'' + y' + y = 0$

10. $y'' - y' + y = 0$

11. $y'' + 8y' + 2y = 0$

12. $y'' + 16y' + 4y = 0$

13. $x^2y'' + y = 0$

14. $x^2y'' + 4y = 0$

15. $x^2y'' + 2xy' + y = 0$

16. $x^2y'' + 8xy' + 4y = 0$

Wronskian. Compute the Wronskian, up a constant multiple, without solving the differential equation.

17. $y'' + y' - xy = 0$

18. $y'' - y' + xy = 0$

19. $2y'' + y' + \sin(x)y = 0$

20. $4y'' - y' + \cos(x)y = 0$

21. $x^2y'' + xy' - y = 0$

22. $x^2y'' - 2xy' + y = 0$

Variation of Parameters. Find the general solution $y_h + y_p$ by applying a variation of parameters formula.

35. $y'' = x^2$

36. $y'' = x^3$

37. $y'' + y = \sin x$

38. $y'' + y = \cos x$

39. $y'' + y' = \ln|x|$

40. $y'' + y' = -\ln|x|$

41. $y'' + 2y' + y = e^{-x}$

42. $y'' - 2y' + y = e^{x}$