5.5 Undetermined Coefficients

The **method of undetermined coefficients** applies to solve differential equations

(1) $$ay'' + by' + cy = r(x).$$

It finds a particular solution y_p *without* the integration steps present in variation of parameters. The method's importance is argued from its direct applicability to second order differential equations in mechanics and circuit theory. The requirements and limitations:

1. The coefficients a, b, c in (1) are constant.
2. The function $r(x)$ is a sum of constants times atoms.

Atoms. An **atom** is a term having one of the forms

$$x^m,\ x^m e^{ax},\ x^m \cos bx,\ x^m \sin bx,\ x^m e^{ax} \cos bx \quad \text{or} \quad x^m e^{ax} \sin bx.$$

The symbols a and b are real constants, with $b > 0$. Symbol $m \ge 0$ is an integer. The terms x^3, $x\cos 2x$, $\sin x$, e^{-x}, $x^6 e^{-\pi x}$ are atoms. Conversely, given $r(x) = 4\sin x + 5xe^x$, then the atoms of $r(x)$ are $\sin x$ and xe^x : split the sum into terms and drop the coefficients 4 and 5 to identify atoms. *Included* as possible functions $r(x)$ in (1) are $\sinh x$ and $\cos^3 x$, due to identities from algebra and trigonometry. *Specifically excluded* are $\ln|x|$, $|x|$, e^{x^2} and fractions like $x/(1+x^2)$.

Atoms A and B are called **related atoms** if their successive derivative formulae contain a common atom. A simple test for non-trigonometric atoms is $A/B = x^p$ for some integer p ($p = 0$, $p > 0$, $p < 0$ allowed). For instance, e^x, xe^x and x^3e^x are related atoms of x^2e^x.

The Basic Trial Solution Method

1. Repeatedly differentiate the atoms of $r(x)$ until no new atoms appear. Multiply the distinct atoms so found by **undetermined coefficients** d_1, d_2, …, d_k, then add to define a **trial solution** y.
2. **Fixup rule**: if the homogeneous equation $ay'' + by' + cy = 0$ has solution y_h containing an atom A which appears in the trial solution y, then replace each **related atom** B in y by xB (other atoms appearing in y are unchanged). Repeat the fixup rule until y contains no atom of y_h. The modified expression y is called the **corrected trial solution**.
3. Substitute y into the differential equation $ay'' + by' + cy = r(x)$. Match atoms left and right to write out linear algebraic equations for the undetermined coefficients d_1, d_2, …, d_k.
4. Solve the equations. The trial solution y with evaluated coefficients d_1, d_2, …, d_k becomes the particular solution y_p.

Superposition. The relation $y = y_h + y_p$ suggests solving equation (1) in two stages: (a) Apply the linear equation **recipe** to find y_h; (b) Apply **the basic trial solution method** to find y_p. We expect to find two arbitrary constants c_1, c_2 in the solution y_h, but in contrast, no arbitrary constants appear in y_p. Calling d_1, d_2, d_3, ... *undetermined coefficients* is misleading, because in fact they are eventually *determined*.

Background. The basic trial solution method requires background in the solution of simultaneous linear algebraic equations, as is often taught in college algebra. Readers with linear algebra background will have an easier time with the details. Literature normally omits details of the basic trial solution method, referencing only *the method of undetermined coefficients*.

Further study. The basic trial solution method is enriched by developing a **library of special methods** for finding y_p, which includes Kümmer's method; see page 256. The library provides a justification of the basic trial solution method. The only background required is college algebra and polynomial calculus . The trademark of the library method is the *absence of linear algebra, tables or special cases*, that can be found in other literature on the subject.

Illustration. Let's solve $y'' - y = x + xe^x$, verifying that $y_h = c_1e^x + c_2e^{-x}$ and $y_p = -x - \frac{1}{4}xe^x + \frac{1}{4}x^2e^x$.

Solution:

Homogeneous solution. The characteristic equation $r^2 - 1 = 0$ has roots $r = \pm 1$. Recipe **case 1** implies $y_h = c_1e^x + c_2e^{-x}$.

Initial trial solution. The atoms of $r(x) = x + xe^x$ are x and xe^x. Differentiation of the atoms gives a new list of distinct atoms 1, x, e^x, xe^x. The undetermined coefficients are given symbols d_1, d_2, d_3, d_4. Then the initial trial solution is

$$y = d_1 + d_2x + d_3e^x + d_4e^x.$$

Fixup rule. The atoms of y_h are e^x and e^{-x}. Atom $A = e^x$ appears in the trial solution y. The **related atoms** of y are e^x and xe^x, which are replaced in y by $x(e^x)$ and $x(xe^x)$, respectively. Then the final trial solution is (the atoms 1 and x were unaffected by the fixup rule):

$$y = d_1 + d_2x + d_3xe^x + d_4x^2e^x.$$

Substitute and find the equations. The details:

$\text{LHS} = y'' - y$	Left side of the equation.
$= [y_1'' - y_1] + [y_2'' - y_2]$	Define $y_1 = d_1 + d_2x$, $y_2 = d_3xe^x + d_4x^2e^x$, then let $y = y_1 + y_2$.
$= [0 - y_1] +$ $[2d_3e^x + 2d_4e^x + 4d_4xe^x]$	Use $y_1'' = 0$ and $y_2'' = 2d_3e^x + 2d_4e^x + 4d_4xe^x + y_2$.
$= -d_1 - d_2x +$ $(2d_3 + 2d_4)e^x + 4d_4xe^x$	Collect on distinct atoms.

Because LHS = RHS and RHS = $x + xe^x$, the above gives the relation

$$-d_1 - d_2 x + (2d_3 + 2d_4)e^x + 4d_4 xe^x = x + xe^x.$$

Equate coefficients of matching atoms left and right to give the system of equations

$$\begin{aligned} 0 &= -d_1, \\ 1 &= -d_2, \\ 0 &= 2d_3 + 2d_4, \\ 1 &= 4d_4. \end{aligned}$$

Solve the equations. Solving by back-substitution gives $d_1 = 0$, $d_2 = -1$, $d_4 = 1/4$, $d_3 = -1/4$. The trial solution with determined coefficients d_1, d_2, d_3, d_4 becomes the particular solution

$$y_p = -x - \frac{1}{4}xe^x + \frac{1}{4}x^2e^x.$$

Report $y = y_h + y_p$. From above, $y_h = c_1e^x + c_2e^{-x}$ and $y_p = -x - \frac{1}{4}xe^x + \frac{1}{4}x^2e^x$. Then $y = y_h + y_p$ is given by

$$y = c_1e^x + c^2e^{-x} - x - \frac{1}{4}xe^x + \frac{1}{4}x^2e^x.$$

Answer check. Computer algebra system `maple` is used.

```
yh:=c1*exp(x)+c2*exp(-x);
yp:=-x-(1/4)*x*exp(x)+(1/4)*x^2*exp(x);
de:=diff(y(x),x,x)-y(x)=x+x*exp(x):
odetest(y(x)=yh+yp,de); # Success for a report of zero.
```

9 Example (Sine–Cosine Trial solution) Verify for $y'' + 4y = \sin x - \cos x$ that $y_p(x) = 5\cos x + 3\sin x$, by using the trial solution $y = A\cos x + B\sin x$.

Solution: Substitute $y = A\cos x + B\sin x$ into the differential equation and use $u'' = -u$ for $u = \sin x$ or $u = \cos x$ to obtain the relation

$$\begin{aligned} \sin x - \cos x &= y'' + 4y \\ &= (-A+4)\cos x + (-B+4)\sin x. \end{aligned}$$

Comparing sides, matching sine and cosine terms, gives

$$\begin{aligned} -A + 4 &= -1, \\ -B + 4 &= 1. \end{aligned}$$

Solving, $A = 5$ and $B = 3$. The trial solution $y = A\cos x + B\sin x$ becomes $y_p(x) = 5\cos x + 3\sin x$. Generally, this method produces linear algebraic equations that must be solved by linear algebra techniques.

10 Example (Basic Trial Solution Method: I)
Solve for y_p in $y'' = 2 - x + x^3$ using the basic trial solution method, verifying $y_p = x^2 - x^3/6 + x^5/20$.

Solution:

Homogeneous solution. The equation $y'' = 0$ has characteristic equation $r^2 = 0$ and therefore $y_h = c_1 + c_2 x$.

Initial trial solution. The right side $r(x) = 2 - x + x^3$ has atoms 1, x, x^3. Repeated differentiation of the atoms gives the new list of atoms 1, x, x^2, x^3. Then the initial trial solution is

$$y = d_1 + d_2 x + d_3 x^2 + d_4 x^3.$$

Fixup rule and final trial solution. The homogeneous solution $y_h = c_1 + c_2 x$ has atoms 1, x. Each of these atoms appears in the initial trial solution. Multiply each related atom in y by x. Repeat the process again to eliminate all duplications. Then the final trial solution is

$$y = x^2(d_1 + d_2 x + d_3 x^2 + d_4 x^3).$$

Substitute and find the equations. The details:

$2 - x + x^3 = y''$	Reverse sides.
$\quad = 2d_1 + 6d_2 x + 12 d_3 x^2 + 20 d_4 x^3$	Substitute the final trial solution.

Equate coefficients of atoms on each side of the equal sign to obtain the system

$$\begin{array}{rcl} 2d_1 &=& 2, \\ 6d_2 &=& -1, \\ 12d_3 &=& 0, \\ 20d_4 &=& 1. \end{array}$$

Solve the equations. This is a triangular system of linear equations for unknowns d_1, d_2, d_3, d_4. Solving gives $d_1 = 1$, $d_2 = -1/6$, $d_3 = 0$, $d_4 = 1/20$.

Report y_p. The expression $y = x^2(d_1 + d_2 x + d_3 x^2 + d_4 x^3)$ after substitution of the values found gives

$$y = x^2(1 - x/6 + x^3/20).$$

11 Example (Basic Trial Solution Method: II)

Solve $y'' + y = 2 + e^x + \sin(x)$ by the basic trial solution method, verifying $y = c_1 \cos(x) + c_2 \sin(x) + 2 + \frac{1}{2}e^x - \frac{1}{2}\sin x$.

Solution:

Homogeneous solution. The characteristic equation $r^2 + 1 = 0$ has roots $r = \pm i$. The recipe **Case 3** implies

$$y_h = c_1 \cos x + c_2 \sin x,$$

where c_1 and c_2 are arbitrary constants. The atoms of y_h are $\cos x$, $\sin x$.

Initial trial solution. The atoms of $r(x) = 2 + e^x + \sin x$ are 1, e^x and $\sin x$. The derivatives of the atoms contain the list of distinct atoms 1, e^x, $\cos x$, $\sin x$. An initial trial solution is

$$y = d_1 + d_2 e^x + d_3 \cos x + d_4 \sin x.$$

Fixup rule. The atoms of y_h appear in the trial solution y. Replace the **related atoms** $\cos x$, $\sin x$ in y by $x \cos x$, $x \sin x$. Then (atoms 1, e^x are unchanged)

$$y = d_1 + d_2 e^x + d_3 x \cos x + d_4 x \sin x.$$

This trial solution has no atoms matching atoms of y_h, therefore the fixup rule finishes with the final trial solution y defined as above.

Substitute and find equations.

$\mathsf{LHS} = y'' + y$	Left side of the equation $y'' + y = 2 + e^x + \sin(x)$.
$= d_1 + 2d_2 e^x - 2d_3 \sin(x) + 2d_4 \cos(x)$	Substitute trial solution y.

The relation LHS = RHS becomes

$$d_1 + 2d_2 e^x - 2d_3 \sin(x) + 2d_4 \cos(x) = 2 + e^x + \sin(x).$$

Equating coefficients of atoms left and right gives the equations

$$\begin{array}{rcl} d_1 &=& 2, \\ 2d_2 &=& 1, \\ -2d_3 &=& 1, \\ 2d_4 &=& 0. \end{array}$$

Solve the equations. There are no details, because the answers are already displayed above as $d_1 = 2$, $d_2 = 1/2$, $d_3 = -1/2$, $d_4 = 0$.

Solution y_p. The particular solution is obtained from the trial solution y by replacing the undetermined coefficients by their values determined above:

$$y_p = 2 + \frac{1}{2} e^x - \frac{1}{2} \sin x.$$

General Solution. Add y_h and y_p to obtain the general solution

$$y = c_1 \cos(x) + c_2 \sin(x) + 2 + \frac{1}{2} e^x - \frac{1}{2} \sin x.$$

Answer check. Computer algebra system `maple` checks the answer as follows.

```
dsolve(diff(y(x),x,x)+y(x)=2+exp(x)+sin(x),y(x));
# y(x) = 2+1/2*exp(x)-1/2*cos(x)*x+_C1*cos(x)+_C2*sin(x)
```

12 Example (Basic Trial Solution Method: III)

Solve for y_p in $y'' - 2y' + y = (1 + x - x^2)e^x$ by the basic trial solution method, verifying that $y_p = (x^2/2 + x^3/6 - x^4/12)e^x$.

Solution:

Homogeneous solution. The characteristic equation $r^2 - 2r + 1 = 0$ for $y'' - 2y' + y = 0$ has a double root $r = 1$. The recipe **Case 2** implies $y_h = c_1e^x + c_2xe^x$, where c_1 and c_2 are arbitrary constants. The atoms of y_h are e^x and xe^x.

Initial trial solution. The derivatives of $r(x) = (1 + x - x^2)e^x$ use the list of distinct atoms e^x, xe^x, x^2e^x. An initial trial solution is

$$y = (d_1 + d_2x + d_3x^2)e^x.$$

Fixup rule and final trial solution. The atoms of y_h appear in the initial trial solution y. The fixup rule implies y should be multiplied twice by x to obtain the final trial solution

$$y = x^2(d_1 + d_2x + d_3x^2)e^x.$$

Substitute and find the equations. Substitute the final trial y solution into $y'' - 2y' + y = (1 + x - x^2)e^x$, in order to find the undetermined coefficients d_1, d_2, d_3. To present the details, let $q(x) = x^2(d_1 + d_2x + d_3x^2)$, then $y = q(x)e^x$ implies

$$\begin{aligned}
\text{LHS} &= y'' - 2y' + y \\
&= [q(x)e^x]'' - 2[q(x)e^x]' + q(x)e^x \\
&= q(x)e^x + 2q'(x)e^x + q''(x)e^x - 2q'(x)e^x - 2q(x)e^x + q(x)e^x \\
&= q''(x)e^x \\
&= [2d_1 + 6d_2x + 12d_2x^2]e^x.
\end{aligned}$$

Matching coefficients of atoms left and right gives the equations

$$\begin{aligned}
2d_1 &= 1, \\
6d_2 &= 1, \\
12d_3 &= -1.
\end{aligned}$$

Solve the equations. The above system needs no further analysis: $d_1 = 1/6$, $d_2 = 1/6$, $d_3 = -1/12$.

Report y_h. The trial solution with evaluated coefficients becomes

$$y_p = x^2(1/2 + x/6 - x^2/12)e^x.$$

General solution. The relation $y = y_h + y_p$ becomes

$$y = c_1e^x + c_2xe^x + x^2(1/2 + x/6 - x^2/12)e^x.$$

Answer check. The `maple` code:

```
de:=diff(y(x),x,x)-2*diff(y(x),x)+y(x)=(1+x-x^2)*exp(x);
dsolve(de,y(x));
# y(x) =1/2*exp(x)*x^2 + 1/6*exp(x)*x^3 -
#       1/12*exp(x)*x^4+_C1*exp(x)+_C2*exp(x)*x
```

Exercises 5.5

Polynomial Solutions. Determine a polynomial solution y_p for the given differential equation.

1. $y'' = x$

2. $y'' = x - 1$

3. $y'' = x^2 - x$

4. $y'' = x^2 + x - 1$

5. $y'' - y' = 1$

6. $y'' - 5y' = 10$

7. $y'' - y' = x$

8. $y'' - y' = x - 1$

9. $y'' - y' + y = 1$

10. $y'' - y' + y = -2$

11. $y'' + y = 1 - x$

12. $y'' + y = 2 + x$

13. $y'' - y = x^2$

14. $y'' - y = x^3$

Polynomial-Exponential Solutions. Determine a solution y_p for the given differential equation.

15. $y'' + y = e^x$

16. $y'' + y = e^{-x}$

17. $y'' = e^{2x}$

18. $y'' = e^{-2x}$

19. $y'' - y = (x+1)e^{2x}$

20. $y'' - y = (x-1)e^{-2x}$

21. $y'' - y' = (x+3)e^{2x}$

22. $y'' - y' = (x-2)e^{-2x}$

23. $y'' - 3y' + 2y = (x^2+3)e^{3x}$

24. $y'' - 3y' + 2y = (x^2-2)e^{-3x}$

Sine and Cosine Solutions. Determine a solution y_p for the given differential equation.

25. $y'' = \sin(x)$

26. $y'' = \cos(x)$

27. $y'' + y = \sin(x)$

28. $y'' + y = \cos(x)$

29. $y'' = (x+1)\sin(x)$

30. $y'' = (x+1)\cos(x)$

31. $y'' - y = (x+1)e^x \sin(2x)$

32. $y'' - y = (x+1)e^x \cos(2x)$

33. $y'' - y' - y = (x^2+x)e^x \sin(2x)$

34. $y'' - y' - y = (x^2+x)e^x \cos(2x)$

Undetermined Coefficients Algorithm. Determine a solution y_p for the given differential equation. These exercises require decomposition into easily-solved equations.

35. $y'' = x + \sin(x)$

36. $y'' = 1 + x + \cos(x)$

37. $y'' + y = x + \sin(x)$

38. $y'' + y = 1 + x + \cos(x)$

39. $y'' + y = \sin(x) + \cos(x)$

40. $y'' + y = \sin(x) - \cos(x)$

41. $y'' = x + xe^x + \sin(x)$

42. $y'' = x - xe^x + \cos(x)$

43. $y'' - y = \sinh(x) + \cos^2(x)$

44. $y'' - y = \cosh(x) + \sin^2(x)$

45. $y'' + y' - y = x^2e^x + xe^x\cos(2x)$

46. $y'' + y' - y = x^2e^{-x} + xe^x\sin(2x)$